ISSN 2075-9827

Карпатські математичні публікації

Carpathian mathematical publications

Карпатские математические публикации

Том 7

№ 1 2015

Карпатські математичні публікації
Науковий журнал
Друкується за рішенням Вченої ради
Прикарпатського національного університету імені Василя Стефаника

Адреса редакції: Факультет математики ...
Прикарпатський національний університет імені Василя Стефаника
76018, Івано-Франківськ, вул. Шевченка, 57
Тел.: (0342) 59-60-50
e-mail: cmp.if.ua@gmail.com
Адреса в інтернеті: http://www.journals.pu.if.ua/index.php/cmp/

Карпатські математичні публікації

Науковий журнал
Т.7, №1
2015

Зміст

Carpathian mathematical publications

SCIENTIFIC JOURNAL
V.7, №1
2015

CONTENTS

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 3–5
doi:10.15330/cmp.7.1.3-5

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.3–5

ATAMANYUK L.S.

WIENER WEIGHTED ALGEBRA OF FUNCTIONS OF INFINITELY MANY VARIABLES

In this article we consider a weighted Wiener type Banach algebra of infinitely many variables. The main result is a description of the spectrum of this algebra.

Key words and phrases: weighted algebra, spectrum of an algebra.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: atamanyuk110@gmail.com

INTRODUCTION

Algebras of functions with absolutely summing Fourier series are called usually as Wiener type algebras. In [1] I. Gohberg, S. Goldberg, M.A. Kaashoek have given a description of a Wiener algebra with weights consisting of functions of several complex variables. They, in particular, described the spectrum (the set of multiplicative linear functionals) of this algebra. In this paper we consider a Wiener weighted algebra $W(w)$ of functions of infinitely many variables. Our main result is Theorem 1, where we have described the spectrum of $W(w)$. Also we consider the algebra $W_+(w)$ which consists of analytic functions on a Cartesian product of balls of radii ρ_{2,k_m}. The spectrum of the algebra $W_+(w)$ may be identified with the Cartesian product of these balls. In the case when $w(k_m) = 1$ for $k_m \geq 0$ these results was obtained by A.V. Zagorodnyuk and M.A. Mitrofanov in [2]. Spectra of algebras of analytic functions on Banach spaces were investigated by many authors in [3, 4, 5, 6]. For more informations on analytic functions on Banach spaces we refer the reader to [7, 8].

1 MAIN RESULTS

Let $c_{00}(\mathbb{Z})$ be a set of finite integer valued sequences $k = (k_\alpha)_{\alpha\in\mathbb{N}} = (k_1,\ldots,k_l,0,\ldots)$, $k_\alpha \in \mathbb{Z}$ for all $\alpha \in \mathbb{N}, |k| = \sum_\alpha |k_\alpha|$. A *weight* is a map $w : c_{00}(\mathbb{Z}) \to [1;\infty)$ satisfying $w(k+s) \leqslant w(k)w(s)$, where $w(k) = w(k_1,\ldots,k_l,0,\ldots)$, $w(s) = w(s_1,\ldots,s_r,0,\ldots)$, $k+s = (k_1+s_1,\ldots,k_n+s_n,0,\ldots)$. Let $W_0(w)$ be the space of all complex valued functions $f : l_\infty \to \mathbb{C}$ of the form $f(x) = \sum\limits_{|k|=0}^{m} a_k e^{i(k,x)} := \sum\limits_{|k|=0}^{m} a_{k_1,\ldots,k_l} e^{i\sum_\alpha k_\alpha x_\alpha}$, $m \in \mathbb{N}$, with the norm

$$\|f\| := \sum_{|k|=0}^{m} |a_k| w(k) = \sum_{|k|=0}^{m} |a_{k_1,\ldots,k_l}| w(k_1,\ldots,k_l,0,\ldots), \tag{1}$$

УДК 517.98
2010 *Mathematics Subject Classification:* 46J15, 46J20.

where i is the imaginary unit. Let us denote by $W(w)$ the completion of $W_0(w)$ with respect to the norm (1). Hence every element f on $W(w)$ has the form

$$f(x) = \sum_{|k|=0}^{\infty} a_k e^{i(k,x)} \tag{2}$$

and

$$\|f\| = \sum_{|k|=0}^{\infty} |a_k| w(k) < \infty. \tag{3}$$

Lemma 1. *Elements of the form* (2) *under condition* (3) *generate a weighted Banach-Wiener algebra.*

Proof. It is easy to see that $W(w)$ is an algebra. Let $f_n = \sum\limits_{|k_1|+\ldots+|k_l|=n} a_{k_1,\ldots,k_l} e^{i\sum\limits_{\alpha} k_\alpha x_\alpha}$, then $\|f_n\| = \sum\limits_{|k_1|+\ldots+|k_l|=n} |a_{k_1,\ldots,k_l}| w(k_1,\ldots,k_l)$. If $f, g \in W(w)$, then

$$\|f_n g_m\| = \sum_{|k|+|s|=n+m} |a_{k_1,\ldots,k_l} b_{s_1,\ldots,s_r}| w(k_1,\ldots,k_l,s_1,\ldots,s_r)$$
$$\leqslant \sum_{|k|+|s|=n+m} |a_k||b_s| w(k) w(s) = \sum_{|k|=n} |a_k| w(k) \sum_{|s|=m} |b_s| w(s) = \|f_n\| \|g_m\|.$$

Thus $\|fg\| = \sum\limits_{n+m=0}^{\infty} \|f_n g_m\| \leqslant \sum\limits_{n+m=0}^{\infty} \|f_n\|\|g_m\| \leqslant \sum\limits_{n=0}^{\infty} \|f_n\| \sum\limits_{m=0}^{\infty} \|g_m\| = \|f\|\|g\|$.

The space $W(w)$ may be identified with the weighted $\ell_{1,w}(\mathbb{Z}^n)$ space of all sequences $\{a_{k_1,\ldots,k_l} w(k_1,\ldots,k_l)\}_{-\infty}^{\infty}$ which are in $\ell_1(\mathbb{Z}^n)$ and hence $W(w)$ is a Banach space. □

We describe multiplicative linear functionals on $W(w)$. Let $w(k_m) = w(0,\ldots,k_m,0,\ldots)$,

$$\rho_{1,k_m} := \sup_{k_m<0} \sqrt[k_m]{w(k_m)} = \lim_{k_m\to-\infty} \sqrt[k_m]{w(k_m)}, \tag{4}$$

$$\rho_{2,k_m} := \inf_{k_m>0} \sqrt[k_m]{w(k_m)} = \lim_{k_m\to\infty} \sqrt[k_m]{w(k_m)}. \tag{5}$$

Then $0 < \rho_{1,k_m} \leqslant \rho_{2,k_m} < \infty$.

For each $\lambda_\alpha, \alpha \in \mathbb{N}$, in the annulus $\rho_{1,k_\alpha} \leqslant |\lambda_\alpha| \leqslant \rho_{2,k_\alpha}$ we define functionals $h_\lambda(f)$ on $W(w)$ as follows

$$h_\lambda(f) = \sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod_\alpha \lambda_\alpha^{k_\alpha},$$

where $\lambda = (\lambda_1,\ldots,\lambda_l,0,\ldots)$. Then (4) and (5) imply that $h_\lambda(f)$ is well-defined. It is easy to see that $h_\lambda(f)$ is multiplicative and linear functional.

Theorem 1. *Each multiplicative linear functional φ on $W(w)$ is an $h_\lambda(f)$ for some λ_α in $\rho_{1,\alpha} \leqslant |\lambda_\alpha| \leqslant \rho_{2,\alpha}$.*

Proof. Let $y_m \in W(w)$ be given by $y_m = e^{ix_m}$. The element y_m is invertible in $W(w)$. For any positive integer k_m, we have $\|y_m^{k_m}\| = w(k_m)$ and $\|y_m^{-k_m}\| = w(-k_m)$. Since $\varphi(y_m^{-k_m}) = (\varphi(y_m))^{-k_m}$, it follows that $w(-k_m)^{-1/k_m} \leqslant |\varphi(y_m)| \leqslant w(k_m)^{1/k_m}$. But then (4) and (5) imply that $\lambda_m := \varphi(y_m)$ belongs to the annulus $\rho_{1,k_m} \leqslant |\lambda_m| \leqslant \rho_{2,k_m}$. Finally, observe that $f = \sum\limits_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} e^{i\sum_\alpha k_\alpha x_\alpha} \in W(w)$ can be written as $f = \sum\limits_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod\limits_\alpha e^{ik_\alpha x_\alpha} = \sum\limits_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod\limits_\alpha y_\alpha^{k_\alpha}$

and the series converges in the norm of $W(w)$. Since φ is continuous, linear and multiplicative functional, we conclude that

$$\varphi(f(x)) = \varphi\Big(\sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod_\alpha y_\alpha^{k_\alpha}\Big) = \sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod_\alpha \varphi(y_\alpha)^{k_\alpha} = \sum_{|k|=0}^{\infty} a_{k_1,\ldots,k_l} \prod_\alpha \lambda_\alpha^{k_\alpha} = h_\lambda(f).$$

□

On other words the spectrum of $W(w)$ may be identified with the set of point evaluation functionals at points $\{(x_1,x_2,\ldots,x_\alpha,\ldots) \in \ell_\infty : \rho_{1,\alpha} \leqslant |x_\alpha| \leqslant \rho_{2,\alpha}\}$.

Remark 1. If we consider the case $w(k_m) = 0$ for $k_m < 0$, then the norm, which has been defined in (3), is actually a seminorm. Taking the quotient algebra with respect to the kernel of this seminorm we will obtain the algebra $W_+(w)$ which consists of analytic functions on a Cartesian product of balls of radii ρ_{2,k_m}. By the same way like in Theorem 1 we can show that the spectrum of $W_+(w)$ coincides with the set of point evaluation functionals at points $\{(x_1,x_2,\ldots,x_\alpha,\ldots) \in \ell_\infty : |x_\alpha| \leqslant \rho_{2,\alpha}\}$.

REFERENCES

[1] Gohberg I., Goldberg S., Kaashoek M.A. Classes of Linear Operators. Birchauser-Verlag, Basel, 1993.

[2] Zagorodnuyk A.V., Mitrofanov M.A. *An analog of Wiener's theorem for infinite-dimensional Banach spaces.* Math. Notes 2015, **97** (1-2), 179–189. doi:10.1134/S00014346150102O4 (translation of Mat. Zametki 2015, 97 (2), 191–202. doi:10.4213/mzm9371 (in Russian))

[3] Alencar R., Aron R., Galindo P., Zagorodnyuk A. *Algebras of symmetric holomorphic functions on ℓ_p.* Bull. London Math. Soc. 2003, **35**, 55–64.

[4] Aron R.M., Cole B.J., Gamelin T.W. *Spectra of algebras of analytic functions on a Banach space.* J. Reine Angew. Math. 1991, **415**, 51–93.

[5] Aron R.M., Galindo P., Garcia D., Maestre M. *Regularity and algebras of analytic functions in infinite dimensions.* Trans. Amer. Math. Soc. 1996, **348**, 543–559.

[6] Zagorodnyuk A. *Spectra of Algebras of Entire Functions on Banach Spaces.* Proc. Amer. Math. Soc. 2006, **134**, 2559–2569.

[7] Lopushansky O., Zagorodnyuk A. Infinite Dimentional Holomorphy. Spectra and Hilbertian Structures. AGH, Krakow, 2013.

[8] Mujica J. Complex Analysis in Banach Spaces. North-Holland, Amsterdam-New York-Oxford, 1986.

Received 10.03.2015

Атаманюк Л.С. *Алгебра Вінера з вагами від нескінченного числа змінних* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 3–5.

Розглянуто зважену банахову алгебру Вінера від нескінченного числа змінних. Основним результатом є опис спектру цієї алгебри.

Ключові слова і фрази: зважена алгебра, спектр алгебри.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 6–21
doi:10.15330/cmp.7.1.6-21

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.6–21

BALKAN Y.S.[1], AKTAN N.[2]

ALMOST KENMOTSU f-MANIFOLDS

In this paper we consider a generalization of almost Kenmotsu f-manifolds. We get basic Riemannian curvature, sectional curvatures and scalar curvature properties of such type manifolds. Finally, we give two examples.

Key words and phrases: f-structure, almost Kenmotsu f-manifolds.

[1] Duzce University, Konuralp Yerleshkesi, 81620, Duzce, Turkey
[2] Necmettin Erbakan University, Science Faculty Dean's Office, Meram Campus, 42090, Konya, Turkey
E-mail: y.selimbalkan@gmail.com (Balkan Y.S.), nesipaktan@gmail.com (Aktan N.)

1 INTRODUCTION

Let M be a real $(2n+s)$-dimensional smooth manifold. M admits f-structure [8] if there exists a non-null smooth $(1,1)$ tensor field φ, tangent bundle TM, satisfying $\varphi^3+\varphi=0$, $rank\,\varphi=2n$. An f-structure is a generalization of almost complex $(s=0)$ and almost contact $(s=1)$ structure. In the latter case M is orientable [9]. Corresponding to two complementary projection operators P and Q applied to TM, defined by $P=-\varphi^2$ and $Q=\varphi^2+I$, where I identity operator, there exist two complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\dim(\mathcal{D})=2n$ and $\dim(\mathcal{D}^{\perp})=s$. The following relations hold [6]

$$\varphi P=P\varphi=\varphi,\quad \varphi Q=Q\varphi=0,\quad \varphi^2P=-P,\quad \varphi^2Q=0.$$

Thus, we have an almost complex distribution $\left(\mathcal{D},\ J=\varphi_{|\mathcal{D}},\ J^2=-I\right)$ and φ acts on $\mathcal{D}^{\perp}$ as a null operator. It follows that

$$TM=\mathcal{D}\oplus\mathcal{D}^{\perp},\quad \mathcal{D}\cap\mathcal{D}^{\perp}=\{0\}.$$

Assume that $\mathcal{D}_p^{\perp}$ is spanned by s globally defined orthonormal vector $\{\xi_i\}$ at each point $p\in M$, $1\le i\le s$, with its dual set $\{\eta^i\}$. Then one obtains

$$\varphi^2=-I+\sum_{i=1}^{s}\eta^i\otimes\xi_i.$$

In the above case, M is called a globally framed manifold (or simply an f-manifold) ([1], [5] and [4]) and we denote its frame structure by $M(\varphi,\xi_i)$. From the above conditions one has

$$\varphi\xi_i=0,\quad \eta^i\circ\varphi=0,\quad \eta^i(\xi_j)=\delta_i^j.$$

УДК 515.165.7
2010 *Mathematics Subject Classification:* 53D10, 53C15, 53C25, 53C35.

Now we consider Riemannian metric g on M that is compatible with an f-structure such that

$$g(\varphi X,Y)+g(X,\varphi Y)=0,\ g(\varphi X,\varphi Y)=g(X,Y)-\sum_{i=1}^{s}\eta^i(X)\eta^i(Y),\ g(X,\xi_i)=\eta^i(X).$$

In the above case, we say that M is a metric f-manifold and its associated structure will be denoted by $M(\varphi,\xi_i,\eta^i,g)$.

A framed structure $M(\varphi,\xi_i)$ is normal [5] if the torsion tensor N_φ of φ is zero i.e., if

$$N_\varphi=N+2\sum_{i=1}^{s}d\eta^i\otimes\xi_i=0,$$

where N denotes the Nijenhuis tensor field of φ.

Define a 2-form Φ on M by $\Phi(X,Y)=g(\varphi X,Y)$, for any $X,Y\in\Gamma(TM)$. The Levi-Civita connection ∇ of a metric f-manifold satisfies the following formula [1]:

$$\begin{aligned}2g((\nabla_X\varphi)Y,Z)&=3d(X,\varphi Y,\varphi Z)-3d(X,Y,Z)\\&\quad+g(N(Y,Z),\varphi X)+N_j^2(Y,Z)\eta^j(X)\\&\quad+2d\eta^j(\varphi Y,X)\eta^j(Z)-2d\eta^j(\varphi Z,X)\eta^j(Y),\end{aligned}$$

where the tensor field N_j^2 is defined by

$$N_j^2(X,Y)=\left(L_{\varphi X}\eta^j\right)Y-\left(L_{\varphi Y}\eta^j\right)X=2d\eta^j(\varphi X,Y)-2d\eta^j(\varphi Y,X),$$

for each $j\in\{1,\dots,s\}$. Following the terminology introduced by Blair [1], we say that a normal metric f-manifold is a K-manifold if its 2-form Φ closed (i.e., $d\Phi=0$). Since $\eta^1\wedge\cdots\wedge\eta^s\wedge\Phi^n\neq0$, a K-manifold is orientable. Furthermore, we say that a K-manifold is a C-manifold if each η^i is closed, an S-manifold if $d\eta^1=d\eta^2\cdots=d\eta^s=\Phi$.

Note that, if $s=1$, namely if M is an almost contact metric manifold, the condition $d\Phi=0$ means that M is quasi-Sasakian. M is said a K-contact manifold if $d\eta=\Phi$ and ξ is Killing.

Falcitelli and Pastore introduced and studied a class of manifolds which is called almost Kenmotsu f-manifold [3]. Such manifolds admit an f-structure with s-dimensional parallelizable kernel. A metric $f.pk$-manifold of dimension $(2n+s)$, $s\ge1$, with $f.pk$-structure (φ,ξ_i,η^i,g), is said to be a almost Kenmotsu $f.pk$-manifold if the 1-forms η^i's are closed and $d\Phi=2\eta^1\wedge\Phi$. Several foliations canonically associated with an almost Kenmotsu $f.pk$-manifold are studied and locally conformal almost Kenmotsu $f.pk$-manifolds are characterized by Falcitelli and Pastore. Öztürk et al. studied almost α-cosymplectic f-manifolds [6].

In this paper we consider a generalization of almost Kenmotsu f-manifolds. We get some curvature properties.

Throughout this paper we use the notations $\overline{\eta}=\eta^1+\eta^2+\cdots+\eta^s$, $\overline{\xi}=\xi_1+\xi_2+\cdots+\xi_s$ and $\overline{\delta}_i^j=\delta_i^1+\delta_i^2+\cdots+\delta_i^s$.

2 ALMOST KENMOTSU f-MANIFOLDS

Almost Kenmotsu f-manifolds firstly defined and studied by Aktan et al. as mentioned below [6].

Definition 2.1 ([6]). *Let $M(\varphi,\xi_i,\eta^i,g)$ be $(2n+s)$-dimensional metric f-manifold. For each η^i, $1\le i\le s$, 1-forms and each Φ 2-forms, if $d\eta^i=0$ and $d\Phi=2\overline{\eta}\wedge\Phi$ satisfy, then M is called almost Kenmotsu f-manifold.*

Let M be an almost Kenmotsu f-manifold. Since the distribution $\mathcal{D}$ is integrable, we have $L_{\xi_i}\eta^j=0$, $[\xi_i,\xi_j]\in\mathcal{D}$ and $[X,\xi_j]\in\mathcal{D}$ for any $X\in\Gamma(\mathcal{D})$. Then the Levi-Civita connection is given by

$$2g\left((\nabla_X\varphi)Y,Z\right)=2\left(\sum_{j=1}^{s}\left(g(\varphi X,Y)\xi_j-\eta^j(Y)\varphi X\right),Z\right)+g\left(N(Y,Z),\varphi X\right), \tag{1}$$

for any $X,Y,Z\in\Gamma(TM)$. Putting $X=\xi_i$ we obtain $\nabla_{\xi_i}\varphi=0$ which implies $\nabla_{\xi_i}\xi_j\in\mathcal{D}^\perp$ and then $\nabla_{\xi_i}\xi_j=\nabla_{\xi_j}\xi_i$, since $[\xi_i,\xi_j]=0$.

We put $A_iX=-\nabla_X\xi_i$ and $h_i=\frac{1}{2}\left(L_{\xi_i}\varphi\right)$, where L denotes the Lie derivative operator.

Proposition 2.1 ([6]). *For any $i\in\{1,\dots,s\}$ the tensor field A_i is a symmetric operator such that*

1) $A_i(\xi_j)=0$, *for any* $j\in\{1,\dots,s\}$,
2) $A_i\circ\varphi+\varphi\circ A_i=-2\varphi$,
3) $tr(A_i)=-2n$.

Proof. Equality $d\eta^i=0$ implies that A_i is symmetric.

1) For any $i,j\in\{1,\dots,s\}$ deriving $g(\xi_i,\xi_j)=\delta_i^j$ with respect to ξ_k, using $\nabla_{\xi_i}\xi_j=\nabla_{\xi_j}\xi_i$, we get $2g(\xi_k,A_i(\xi_j))=0$. Since $\nabla_{\xi_i}\xi_j\in\mathcal{D}^\perp$, we conclude $A_i(\xi_j)=0$.

2) For any $Z\in\Gamma(TM)$, we have $\varphi(N(\xi_i,Z))=(L_{\xi_i}\varphi)Z$ and, on the other hand, since $\nabla_{\xi_i}\varphi=0$,

$$L_{\xi_i}\varphi=A_i\circ\varphi-\varphi\circ A_i. \tag{2}$$

One can easily obtain from (2)

$$-A_iX=-\varphi^2X-\varphi h_iX. \tag{3}$$

Applying (1) with $Y=\xi_i$, we have

$$2g(\varphi A_iX,Z)=-2g(\varphi X,Z)-g(\varphi N(\xi_i,Z),X),$$

which implies the desired result.

3) Considering local adapted orthonormal frame $\{X_1,\dots,X_n,\varphi X_1,\dots,\varphi X_n,\xi_1,\dots\xi_s\}$, by 1) and 2), one has

$$trA_i=\sum_{j=1}^{n}\left(g(A_iX_j,X_j)+g(A_i\varphi X_j,\varphi X_j)\right)=-2\sum_{j=1}^{n}g(\varphi X_j,\varphi X_j)=-2n.$$

□

Proposition 2.2 ([1]). *For any $i\in\{1,\dots,s\}$ the tensor field h_i is a symmetric operator and satisfies*

i) $h_i\xi_j=0$, *for any* $j\in\{1,\dots,s\}$,
ii) $h_i\circ\varphi+\varphi\circ h_i=0$,
iii) $trh_i=0$,
iv) $tr\varphi h_i=0$.

Proposition 2.3. *∇_φ satisfies the following relation [6]:*

$$(\nabla_{X\varphi})Y+(\nabla_{\varphi X}\varphi)\varphi Y=\sum_{i=1}^{s}\left[-\left(\eta^i(Y)\varphi X+2g(X,\varphi Y)\xi_i\right)-\eta^i(Y)h_iX\right]. \tag{4}$$

Proof. By direct computations, we get

$$\varphi N(X,Y)+N(\varphi X,Y)=2\sum_{i=1}^{s}\eta^i(X)h_iY,$$

and

$$\eta^i(N(\varphi X,Y))=0.$$

From (1) and the equations above, the proof is completed. □

3 ALMOST KENMOTSU f-MANIFOLDS WITH ξ BELONGING TO THE (κ,μ,ν)-NULLITY DISTRIBUTION

Definition 3.1. *Let M be an almost Kenmotsu f-manifold, κ,μ and ν are real constants. We say that M verifies the (κ,μ,ν)-nullity condition if and only if for each $i\in\{1,\dots,s\}$, $X,Y\in\Gamma(TM)$ the following identity holds*

$$\begin{aligned}R(X,Y)\xi_i&=\kappa\left(\overline{\eta}(X)\varphi^2Y-\overline{\eta}(Y)\varphi^2X\right)+\mu\left(\overline{\eta}(Y)h_iX-\overline{\eta}(X)h_iY\right)\\&\quad+\nu\left(\overline{\eta}(Y)\varphi h_iX-\overline{\eta}(X)\varphi h_iY\right).\end{aligned} \tag{5}$$

Lemma 3.1. *Let M be an almost Kenmotsu f-manifold verifiying (κ,μ,ν)-nullity condition. Then we have:*

(i) $h_i\circ h_j=h_j\circ h_i$ *for each* $i,j\in\{1,2,\dots,s\}$,
(ii) $\kappa\le-1$,
(iii) *if* $\kappa<-1$ *then, for each* $i\in\{1,2,\dots,s\}$, h_i *has eigenvalues* $0,\pm\sqrt{-(\kappa+1)}$.

Proof. From (5) it follows that for each $X\in\Gamma(TM)$, $i,j\in\{1,2,\dots,s\}$

$$R(\xi_j,X)\xi_i-\varphi R(\xi_j,\varphi X)\xi_i=2\kappa\varphi^2X.$$

Using

$$R(\xi_j,X)\xi_i-\varphi R(\xi_j,\varphi X)\xi_i=2\left[-\varphi^2X+(h_i\circ h_j)X\right]$$

we obtain

$$(h_i\circ h_j)X=(\kappa+1)\varphi^2X=(h_j\circ h_i)X \tag{6}$$

and then (i) is verified. Next from (6) we get

$$h_i^2X=(\kappa+1)\varphi^2X, \tag{7}$$

$$h_i^2X=-(\kappa+1)X,\quad X\in\Gamma(\mathcal{D}). \tag{8}$$

Then, using Proposition 2 and (8) we obtain that the eigenvalues of h_i^2 are 0 and $-(\kappa+1)$. Moreover h_i is symmetric: $\|h_iX\|^2=-(\kappa+1)\|X\|^2$. Hence $\kappa\le-1$. Finally let t be a real eigenvalue of h_i and X be an eigenvector corresponding to t. Then $t^2\|X\|^2=-(\kappa+1)\|X\|^2$ and $t=\pm\sqrt{-(\kappa+1)}$. Taking Proposition 2 into account we get (iii). □

Proposition 3.1. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition. Then*
$$h_1 = \cdots = h_s. \tag{9}$$

Proof. If $\kappa = -1$ then from (7) and the symmetry of each h_i we have $h_1 = \cdots = h_s = 0$. Let now $\kappa < -1$. We fix $x \in M$ and $i \in \{1,2,\ldots,s\}$. Since h_i is symmetric then we have $\mathcal{D}_x = (\mathcal{D}_+)_x \oplus (\mathcal{D}_-)_x$, where $(\mathcal{D}_+)_x$ is the eigenspace of h_i corresponding to the eigenvalue $\lambda = \sqrt{-(\kappa+1)}$ and $(\mathcal{D}_-)_x$ is the eigenspace of h_i corresponding to the eigenvalue $-\lambda$. If $X \in \mathcal{D}_x$ then we can write $X = X_+ + X_-$, where $X_+ \in (\mathcal{D}_+)_x$, $X_- \in (\mathcal{D}_-)_x$ so that $h_i X = \lambda(X_+ + X_-)$. We fix $j \in \{1,2,\ldots,s\}$, $j \neq i$. Then from (6) we get $h_j X = h_j(X_+ + X_-) = h_j(\frac{1}{\lambda}h_i X_+ - \frac{1}{\lambda}h_i X_-) = \frac{1}{\lambda}(h_j \circ h_i)(X_+ + X_-) = \lambda(X_+ + X_-) = h_i X$. Taking into account Proposition 2 we obtain (9). □

Remark 3.1. *Throughout the paper whenever* (5) *holds we put $h := h_1 = \cdots = h_s$. Then* (5) *becomes*

$$R(X,Y)\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y)hX - \overline{\eta}(X)hY\right) + \nu\left(\overline{\eta}(Y)\varphi hX - \overline{\eta}(X)\varphi hY\right). \tag{10}$$

Furthermore, using (10)*, the symmetry properties of the curvature tensor and the symmetry of φ^2 and h, we get*

$$R(\xi_i,X)Y = \kappa\left(\overline{\eta}(Y)\varphi^2 X - g\left(X,\varphi^2 Y\right)\overline{\xi}\right) + \mu\left(g(X,hY)\overline{\xi} - \overline{\eta}(Y)hX\right) + \nu\left(g(\varphi hX,Y)\overline{\xi} - \overline{\eta}(Y)\varphi hX\right). \tag{11}$$

Remark 3.2. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition, with $\kappa \neq -1$. We denote by $\mathcal{D}_+$ and $\mathcal{D}_-$ the n-dimensional distributions of the eigenspaces of $\lambda = \sqrt{-(\kappa+1)}$ and $-\lambda$, respectively. We have that $\mathcal{D}_+$ and $\mathcal{D}_-$ are mutually orthogonal. Moreover, since φ anticommutes with h, we have $\varphi(\mathcal{D}_+) = \mathcal{D}_-$ and $\varphi(\mathcal{D}_-) = \mathcal{D}_+$. In other words, $\mathcal{D}_+$ is a Legendrian distribution and $\mathcal{D}_-$ is the conjugate Legendrian distribution of $\mathcal{D}_+$.*

Proposition 3.2. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition. Then M is a Kenmotsu f-manifold if and only if $\kappa = -1$.*

Proof. We observed in the proof of Proposition 3.1 that if $\kappa = -1$ then $h = 0$. It follows that (10) reduces to $R(X,Y)\xi_i = \overline{\eta}(Y)\varphi^2 X - \overline{\eta}(X)\varphi^2 Y$. From [2, Proposition 3.4, Theorem 4.3] we get the claim. □

4 Properties Of The Curvature

Let $M(\varphi,\xi_i,\eta^i,g)$ be a $(2n+s)$-dimensional almost Kenmotsu f-manifold. We consider the $(1,1)$-tensor fields defined by
$$l_{ij}(.) = R_{.\xi_i}\xi_j$$
for each $i,j \in \{1,\ldots,s\}$ and put $l_i = l_{ii}$.

Lemma 4.1. *For each $i,j,k \in \{1,\ldots,s\}$ the following identities hold:*
$$\varphi \circ l_{ji} \circ \varphi - l_{ji} = 2\left[h_i \circ h_j - \varphi^2\right], \tag{12}$$
$$\eta_k \circ l_{ji} = 0, \tag{13}$$
$$l_{ji}(\xi_k) = 0, \tag{14}$$
$$\nabla_{\xi_j} h_i = -\varphi \circ l_{ji} - \varphi - (h_j + h_i) - \varphi \circ h_i \circ h_j, \tag{15}$$
$$\nabla_{\xi_i} h_i = -\varphi \circ l_{ji} - \varphi - 2h_i - \varphi h_i^2. \tag{16}$$

Proof. Identity (12) is a rewriting of [7, (3.4)]. Formulas (13) and (14) are an immediate consequence of (12). Next from (3) and $\eta_l \circ (\nabla_{\xi_i} h_k) = 0$ we get
$$l_{ij} = \left(\varphi\left(\nabla_{\xi_j} h_i\right) + \varphi^2 + \varphi \circ h_i + \varphi \circ h_j - h_j \circ h_i\right).$$
Applying φ to both sides we get
$$\left(\nabla_{\xi_j} h_i\right) = \left(-\varphi \circ l_{ij} - \varphi - h_i - h_j - \varphi \circ h_j \circ h_i\right),$$
from which it follows (15). Finally, identity (16) is (15) when $i = j$. □

Remark 4.1. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition. Then for each $i,j \in \{1,\ldots,s\}$ we have*
$$l_{ji} = -\kappa\varphi^2 + \mu h + \nu\varphi h. \tag{17}$$

It follows that all the l_{ji}'s coincide. We put $l = l_{ji}$.

Lemma 4.2. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition. Then for each $i \in \{1,\ldots,s\}$, the following identities hold:*
$$\nabla_{\xi_i} h = -\mu\varphi h + \nu h - 2h, \tag{18}$$
$$l\varphi - \varphi l = 2\mu h\varphi + 2\nu h, \tag{19}$$
$$l\varphi + \varphi l = 2\kappa\varphi, \tag{20}$$
$$Q\xi_i = 2n\kappa\overline{\xi}. \tag{21}$$

Proof. From (16), using (17), we obtain (18). Identities (19) and (20) follow directly from (17) using $h \circ \varphi = -\varphi \circ h$. For the proof of (21) we fix $x \in M$ and $\{E_1,\ldots,E_{2n+s}\}$ a local φ-basis around x with $E_{2n+1} = \xi_1,\ldots,E_{2n+s} = \xi_s$. Then using (11) and $trace(h) = 0$ we get $Q\xi_i = \sum_{j=1}^{2n} R_{\xi_i E_j} E_j = \sum_{j=1}^{2n} \kappa g\left(\varphi^2 E_j, E_j\right)\overline{\xi} = \kappa \sum_{j=1}^{2n} \delta_{jj}\overline{\xi}$. □

Lemma 4.3. *Let $(M,\varphi,\xi_i,\eta_j,g)$ be a $(2n+s)$-dimensional almost Kenmotsu f-manifold. Then the curvature tensor satisfies the identities*
$$g\left(R_{\xi_i X}Y,Z\right) = \sum_{j=1}^{s}\eta_j(Z)\,g\left(\varphi^2 Y,X\right) - \sum_{j=1}^{s}\eta_j(Y)\,g\left(\varphi^2 Z,X\right) + \sum_{j=1}^{s}\eta_j(Z)\,g\left(\varphi h_j Y,X\right) - \sum_{j=1}^{s}\eta_j(Y)\,g\left(\varphi h_j Z,X\right) + g\left((\nabla_Z \varphi h_i)Y - (\nabla_Y \varphi h_i)Z, X\right) \tag{22}$$
and
$$g\left(R_{\xi_i X}Y,Z\right) - g\left(R_{\xi_i X}\varphi Y,\varphi Z\right) + g\left(R_{\xi_i \varphi X}Y,\varphi Z\right) + g\left(R_{\xi_i \varphi X}\varphi Y,Z\right) = 2g\left((\nabla_{h_i X}\varphi)Y,Z\right) + 2\overline{\eta}(Z)\,g\left(h_i X - \varphi X,\varphi Y\right) - 2\overline{\eta}(Y)\,g\left(h_i X - \varphi X,\varphi Z\right) \tag{23}$$
for each $i = 1,\ldots,s$ and $X,Y,Z \in \Gamma(TM)$.

Proof. Using the Riemannian curvature tensor and (8), we obtain (22).

We introduce the operators A and B_i, $i \in \{1,\dots,s\}$, defined by

$$A(X,Y,Z) := 2\overline{\eta}(Y)\, g(\varphi X, \varphi Z) - 2\overline{\eta}(Z)\, g(\varphi X, \varphi Y)$$

and

$$\begin{aligned} B_i(X,Y,Z) := & -g(\varphi X, (\nabla_Y(\varphi \circ h_i))(\varphi Z)) - g(\varphi X, (\nabla_Y(\varphi \circ h_i))Z) \\ & - g(X, (\nabla_Y(\varphi \circ h_i))Z) + g(X, (\nabla_{\varphi Y}(\varphi \circ h_i))(\varphi Z)) \end{aligned}$$

for each $X, Y, Z \in \Gamma(TM)$. By a direct computation and using (22) we get that the left hand side of (23) equals $A(X,Y,Z) + B_i(X,Y,Z) - B_i(X,Z,Y)$. Since

$$\eta_j((\nabla_{\varphi Y} h_i) Z) = \eta_j(\nabla_{\varphi Y}(h_i Z)),$$

we can write

$$\begin{aligned} B_i(X,Y,Z) = & -g(X, (\nabla_Y(\varphi \circ h_i)Z)) + g(X, (\varphi \circ h_i)(\nabla_Y Z)) \\ & + g(X, (\nabla_{\varphi Y}(\varphi \circ h_i \circ \varphi)Z)) + g(X, (\varphi \circ h_i)(\nabla_{\varphi Y}\varphi Z)) \\ & - g(\varphi X, (\nabla_Y(\varphi \circ h_i \circ \varphi)Z)) + g(\varphi X, (\varphi \circ h_i)(\nabla_Y(\varphi Z))) \\ & - g(\varphi X, (\nabla_{\varphi Y}(\varphi \circ h_i)Z)) + g(\varphi X, (\varphi \circ h_i)(\nabla_{\varphi Y}(h_i Z))) \\ = & -g(X, (\nabla_Y \varphi)(h_i Z)) + g(X, h_i((\nabla_Y \varphi)Z)) \\ & + g(X, (h_i \circ \varphi)((\nabla_{\varphi Y}\varphi)Z)) + g(X, \varphi((\nabla_{\varphi Y}\varphi)(h_i Z))) \\ & + \sum_{j=1}^{s} \eta_j((\nabla_{\varphi Y} h_i)Z)\, \eta_j(X). \end{aligned} \tag{24}$$

Moreover, from (3), (4) and Proposition 1 it follows that

$$\begin{aligned} (\varphi \circ (\nabla_{\varphi X}\varphi))Y = & \left(\nabla_{\varphi X}\varphi^2\right)Y - (\nabla_{\varphi X}\varphi)(\varphi Y) \\ = & \sum_{j=1}^{s}((\nabla_{\varphi X}\eta_j)Y\xi_j) + \sum_{j=1}^{s}(\eta_j(Y)\nabla_{\varphi X}\xi_j) \\ & - (\nabla_{\varphi X}\varphi)(\varphi Y) = \sum_{j=1}^{s}((\nabla_{\varphi X})(g(\xi_j, Y))\xi \\ & - g(\nabla_{\varphi X}Y, \xi_j)\xi_j) + \sum_{j=1}^{s}\eta_j(Y)(\varphi X - h_j X) \\ & + \sum_{j=1}^{s}\eta_j(Y) h_j X + \overline{\eta}(Y)\varphi X + 2g(X, \varphi Y)\overline{\xi} + (\nabla_X \varphi)Y. \end{aligned}$$

Hence

$$\begin{aligned} (\varphi \circ (\nabla_{\varphi X}\varphi))Y = & \sum_{j=1}^{s} g(X, \varphi Y)\xi_j - \sum_{j=1}^{s} g(Y, h_j X)\xi_j \\ & + 2\sum_{j=1}^{s}\eta_j(Y)\varphi X + (\nabla_X \varphi)Y. \end{aligned}$$

Furthermore, from (4), for each $j \in \{1,\dots,s\}$ we have

$$\begin{aligned} \eta_i((\nabla_{\varphi Y} h_j)Z) = & \eta_i(\nabla_{\varphi Y}(h_j Z)) = (\nabla_{\varphi Y}\eta_i)(h_j Z) \\ = & -g(h_j Z, \nabla_{\varphi Y}\xi_i) = g(h_j Z, h_i Y - \varphi Y). \end{aligned} \tag{25}$$

Then, using (24) and (25), we get

$$\begin{aligned} B_i(X,Y,Z) = & -g(X, (\nabla_Y \varphi)(h_i Z)) + g(X, h_i((\nabla_Y \varphi)Z)) + 2\overline{\eta}(Z)\, g(h_i X, \varphi Y) \\ & + g(h_i X, (\nabla_Y \varphi)Z) + \overline{\eta}(X)\, g(Y, \varphi h_i Z) - \sum_{j=1}^{s}\eta_j(X)\, g(h_i Z, h_j Y) \\ & + \sum_{j=1}^{s}\eta_j(X)\, g(h_i Z, h_j Y) + g(X, (\nabla_Y \varphi)(h_i Z)) - \overline{\eta}(X)\, g(\varphi Y, h_i Z) \\ = & 2(g(h_i X, (\nabla_Y \varphi)Z) + \overline{\eta}(Z)\, g(h_i X, \varphi Y) - \overline{\eta}(X)\, g(\varphi Y, h_i Z)). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & A(X,Y,Z) + B_i(X,Y,Z) - B_i(X,Z,Y) \\ & = 2(\nabla_Y \Phi)(h_i X, Z) - 2(\nabla_Z \Phi)(h_i X, Y) \\ & + 2\overline{\eta}(Z)\, g(h_i X - \varphi X, \varphi Y) - 2\overline{\eta}(Y)\, g(h_i X - \varphi X, \varphi Z) \end{aligned}$$

and hence (23) follows. □

Remark 4.2. *Let M be an almost Kenmotsu f-manifold. Then from (23) using $(\nabla_{h_i X}\Phi)(Y,Z) = -g((\nabla_{h_i X}\varphi)Y, Z)$, for each $X, Y, Z \in \Gamma(TM)$, we get*

$$\begin{aligned} (\nabla_{h_i X}\varphi)Y = & \frac{1}{2}(\varphi R_{\xi_i \varphi X}Y - R_{\xi_i \varphi X}\varphi Y - \varphi R_{\xi_i X}\varphi Y - R_{\xi_i X}Y) \\ & + g(h_i X - \varphi X, \varphi Y)\overline{\xi} + \overline{\eta}(Y)\left(\varphi h_i X - \varphi^2 X\right). \end{aligned} \tag{26}$$

Lemma 4.4. *Let M be an almost Kenmotsu f-manifold verifying (κ, μ, v)-nullity condition. Then the following identities hold:*

$$(\nabla_X \varphi)Y = g(\varphi X + hX, Y)\xi - \eta(Y)(\varphi X + hX), \tag{27}$$

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X = & (\kappa+1)(\eta(Y)\varphi X - \eta(X)\varphi Y + 2g(\varphi X, Y)\xi) \\ & + \mu(\eta(Y)\varphi hX - \eta(X)\varphi hY) + (1-v)(\eta(Y)hX - \eta(X)hY). \end{aligned} \tag{28}$$

Proof. From (26) we obtain

$$(\nabla_{hX}\varphi)Y = -(\kappa+1)\, g(X,Y)\overline{\xi} + (\kappa+1)\overline{\eta}(Y)X + \overline{\eta}(Y)\varphi hX + g(hX, \varphi Y)\overline{\xi}.$$

Here we replace X with hX and by a direct computation, taking into account (3), (7), we get (27). From (27), since h and φ^2 are self-adjoint, we have

$$(\nabla_X(\varphi \circ h))Y - (\nabla_Y(\varphi \circ h))X = \varphi((\nabla_X h)Y - (\nabla_Y h)X).$$

It follows that for each $Z \in \Gamma(TM)$

$$\begin{aligned} g(R_{XY}\xi_i, Z) = & \overline{\eta}(Y)\, g\left(\varphi^2 X + \varphi hX, Z\right) - \overline{\eta}(X)\, g\left(\varphi^2 Y + \varphi hY, Z\right) \\ & + g(\varphi((\nabla_Y h)X - (\nabla_X h)Y), Z), \end{aligned} \tag{29}$$

where we use (5) of [6] and (27). From (29) and the symmetry of h and φ^2 it follows that

$$\varphi\left(\left(\nabla_Y h\right)X-\left(\nabla_X h\right)Y\right)=R_{XY}\xi_i-\overline{\eta}\left(Y\right)\left(\varphi^2X+\varphi hX\right)+\overline{\eta}\left(X\right)\left(\varphi^2Y+\varphi hY\right).$$

Then, applying φ to both sides of the last identity, using (10) and

$$\eta_l\left(\left(\nabla_Y h\right)X-\left(\nabla_X h\right)Y\right)=-2\left(\kappa+1\right)g\left(\varphi X,Y\right),\quad l\in\{1,\ldots,s\},$$

we get (28). □

Theorem 1. *Let* Z=$(M,\varphi,\xi_i,\eta_j,g)$ *be a* $(2n+s)$*-dimensional almost Kenmotsu* f*-manifold and* $\left(\widetilde{\varphi},\widetilde{\xi}_i,\widetilde{\eta}_j,\widetilde{g}\right)$ *be an almost* f*-structure on* M *obtained by a* $\mathcal{D}$*-homothetic transformation of constant* α*. If* Z *verifies the* (κ,μ,ν)*-nullity condition for certain real constants* (κ,μ,ν) *then* $\left(M,\widetilde{\varphi},\widetilde{\xi}_i,\widetilde{\eta}_j,\widetilde{g}\right)$ *verifies the* $(\widetilde{\kappa},\widetilde{\mu},\widetilde{\nu})$*-nullity condition, where*

$$\widetilde{\kappa}=\frac{\kappa}{\alpha},\quad\widetilde{\mu}=\frac{\mu}{\alpha},\quad\widetilde{\nu}=\frac{\nu}{\alpha}.$$

Proof. From (18) and (9) it follows that $\widetilde{h}_1=\cdots=\widetilde{h}_s$. Then, using (27), by a direct calculation we get the claim. □

Lemma 4.5. *Let* M *be an almost Kenmotsu* f*-manifold verifying the* (κ,μ,ν)*-nullity condition. Then*

$$X,Y\in\Gamma\left(\mathcal{D}_+\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_+\right),\tag{30}$$

$$X,Y\in\Gamma\left(\mathcal{D}_-\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_-\right),\tag{31}$$

$$X\in\Gamma\left(\mathcal{D}_+\right),\ Y\in\Gamma\left(\mathcal{D}_-\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_-\oplus\ker\left(\varphi\right)\right),\tag{32}$$

$$X\in\Gamma\left(\mathcal{D}_-\right),\ Y\in\Gamma\left(\mathcal{D}_+\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_+\oplus\ker\left(\varphi\right)\right).\tag{33}$$

Proof. From (28) we get $g\left(\left(\nabla_Xh\right)\varphi Z-\left(\nabla_{\varphi Z}h\right)X,Y\right)=0$, for each $X,Y,Z\in\Gamma\left(\mathcal{D}_+\right)$. On the other hand, since h is symmetric, from Remark 2 we have

$$g\left(\left(\nabla_Xh\right)\varphi Z-\left(\nabla_{\varphi Z}h\right)X,Y\right)=-2\lambda g\left(\nabla_X\left(\varphi Z\right),Y\right).$$

Then

$$g\left(\varphi Z,\nabla_XY\right)=-g\left(\nabla_X\left(\varphi Z\right),Y\right),$$

i.e. ∇_XY is normal to $\mathcal{D}_-$. Moreover from (3) and Remark 2 it follows that, for each $i\in\{1,\ldots,s\}$, $g\left(\nabla_XY,\xi_i\right)=-g\left(Y,\nabla_X\xi_i\right)=0$. Then we have (30). The proof of (31) is analogous. If $X\in\Gamma\left(\mathcal{D}_+\right)$, $Y\in\Gamma\left(\mathcal{D}_-\right)$ then from (30) and Remark 2 we get that for each $Z\in\Gamma\left(\mathcal{D}_+\right)$ $g\left(\nabla_XY,Z\right)=-g\left(Y,\nabla_XZ\right)=0$ and then we have (32). Analogously we prove (33). □

Remark 4.3. *It follows from (30) and (31) that* $\mathcal{D}_\pm$ *define two orthogonal totally geodesic Legendrian foliations* $F_\pm$ *on* M.

Lemma 4.6. *Let* M *be an almost Kenmotsu* f*-manifold verifying the* (κ,μ,ν)*-nullity condition. Then for each* $X,Y\in\Gamma(TM)$ *we have*

$$\begin{aligned}\left(\nabla_Xh\right)Y=&\left(\kappa+1\right)g\left(\varphi X,Y\right)\overline{\xi}-g\left(hX,Y\right)\overline{\xi}-\eta\left(Y\right)h\left(X+h\varphi X\right)\\&-\mu\eta\left(X\right)\varphi hY+\left(\nu-2\right)\eta\left(X\right)hY.\end{aligned}\tag{34}$$

Proof. Let be $X,Y\in\Gamma\left(\mathcal{D}\right)$. From Proposition 2, *i)* we get $g\left(h_iY,\xi_j\right)=0$. Taking the derivative of this equality of the direction X we obtain

$$\left(\nabla_Xh\right)Y=-g\left(Y,h_iX+h_i^2\varphi X\right)\xi_j.$$

Then, we write any vector field X on M as $X=X_++\eta_i(X)\xi_j$, X_+ denoting positive component of X in $\mathcal{D}$, and, using (18) and (8), we have

$$\begin{aligned}\left(\nabla_Xh\right)Y=&\left(\nabla_{X_+}h\right)Y_++\overline{\eta}\left(Y\right)\left(\nabla_{X_+}h\right)\overline{\xi}+\overline{\eta}\left(X\right)\left(-\mu\varphi h+\nu h-2h\right)Y\\&-g\left(Y,hX+h^2\varphi X\right)\overline{\xi}-\overline{\eta}\left(Y\right)\left(hX+h^2\varphi X\right)+\overline{\eta}\left(X\right)\left(-\mu\varphi hY+\nu hY-2hY\right).\end{aligned}$$

□

Remark 4.4. *Let* M *be an almost Kenmotsu* f*-manifold verifying the* (κ,μ,ν)*-nullity condition. Then using (27), (34) and (8) we get, for all* $X,Y\in\Gamma(TM)$

$$\begin{aligned}\left(\nabla_X\varphi h\right)Y=&\left(\kappa+1\right)g\left(\varphi^2X,Y\right)\overline{\xi}+g\left(\varphi X,hY\right)\overline{\xi}-\overline{\eta}\left(Y\right)\varphi hX\\&+\left(\kappa+1\right)\overline{\eta}\left(Y\right)\varphi^2X+\mu\overline{\eta}\left(X\right)hY+\left(\nu-2\right)\eta\left(X\right)\varphi hY.\end{aligned}\tag{35}$$

Lemma 4.7. *Let* M *be an almost Kenmotsu* f*-manifold verifying the* (κ,μ,ν)*-nullity condition. Then for each* $X,Y,Z\in\Gamma\left(\mathcal{D}\right)$ *we have*

$$\begin{aligned}R_{XY}hZ-hR_{XY}Z=&s[\kappa\{g\left(Z,\varphi Y\right)X-g\left(Z,\varphi Y\right)\varphi hX-g\left(Z,\varphi X\right)Y+g\left(Z,\varphi X\right)\varphi hY\\&+g\left(Z,X\right)\varphi Y-g\left(Z,\varphi hX\right)\varphi Y-g\left(Z,Y\right)\varphi X+g\left(Z,\varphi hY\right)\varphi X\}\\&+g\left(Z,\varphi Y\right)X-g\left(Z,\varphi Y\right)\varphi hX-g\left(Z,\varphi X\right)Y+g\left(Z,\varphi X\right)\varphi hY\\&-g\left(Z,hY\right)X+g\left(Z,hY\right)\varphi hX+g\left(Z,hX\right)Y-g\left(Z,hX\right)\varphi hY\\&-g\left(Z,X\right)hY+g\left(Z,X\right)\varphi Y+g\left(Z,\varphi hX\right)hY-g\left(Z,\varphi hX\right)\varphi Y\\&+g\left(Z,Y\right)hX-g\left(Z,Y\right)\varphi X-g\left(Z,\varphi hY\right)hX+g\left(Z,\varphi hY\right)\varphi X].\end{aligned}\tag{36}$$

Proof. Let $X,Y,Z\in\Gamma(TM)$. Then by a direct computation we get

$$\begin{aligned}\left(\nabla_X\nabla_Yh\right)Z=&\left(\kappa+1\right)[g\left(\nabla_XZ,\varphi Y\right)\overline{\xi}+g\left(Z,\left(\nabla_X\varphi\right)Y\right)\overline{\xi}+g\left(Z,\varphi\left(\nabla_XY\right)\right)\overline{\xi}\\&+g\left(Z,\varphi Y\right)\left(-\varphi^2X-\varphi hX\right)]-g\left(\nabla_XZ,hY\right)\overline{\xi}-g\left(Z,\left(\nabla_Xh\right)Y\right)\overline{\xi}\\&-g\left(Z,h\left(\nabla_XY\right)\right)\overline{\xi}+g\left(Z,hY\right)\left(\varphi^2X+\varphi hX\right)-g\left(\nabla_XZ,\overline{\xi}\right)\left(hY+h^2\varphi Y\right)\\&-g\left(Z,\nabla_X\overline{\xi}\right)\left(hY+h^2\varphi Y\right)-\overline{\eta}\left(Z\right)\left(\nabla_Xh\right)Y-\overline{\eta}\left(Z\right)h\left(\nabla_XY\right)\\&\left(\kappa+1\right)[\overline{\eta}\left(Z\right)\left(\nabla_X\varphi\right)Y+\overline{\eta}\left(Z\right)\varphi\left(\nabla_XY\right)]-\mu[g\left(\nabla_XY,\overline{\xi}\right)\varphi hZ\\&-g\left(Y,\nabla_X\overline{\xi}\right)\varphi hZ-\overline{\eta}\left(Y\right)\left(\nabla_X\varphi h\right)Z-\overline{\eta}\left(Y\right)\varphi h\left(\nabla_XZ\right)]\\&+\left(\nu-2\right)[g\left(\nabla_XY,\overline{\xi}\right)hZ+g\left(Y,\nabla_X\overline{\xi}\right)hZ+\overline{\eta}\left(Y\right)\left(\nabla_Xh\right)Z+\overline{\eta}\left(Y\right)h\left(\nabla_XZ\right)],\end{aligned}$$

where we used (34), (8) and the antisymmetry of $\nabla_X\varphi$. Hence, using the Ricci identity

$$R_{XY}hZ-hR_{XY}Z=\left(\nabla_X\nabla_Yh\right)Z-\left(\nabla_Y\nabla_Xh\right)Z-\left(\nabla_{[X,Y]}h\right)Z,$$

formulas (34) and (3), the symmetry of $\nabla_X (h \circ \varphi)$, we obtain

$$\begin{aligned} R_{XY}hZ - hR_{XY}Z = {} & (\kappa+1)\left[g\left(Z,(\nabla_X\varphi)Y - (\nabla_Y\varphi)X\right)\overline{\xi} - g(Z,\varphi Y)\left(\varphi^2X + \varphi hX\right)\right. \\ & \left.+ g(Z,\varphi X)\left(\varphi^2Y + \varphi hY\right)\right] - g\left(Z,(\nabla_Xh)Y - (\nabla_Yh)X\right)\overline{\xi} + g(Z,hY)\left(\varphi^2X + \varphi hX\right) \\ & - g(Z,hX)\left(\varphi^2Y + \varphi hY\right) - g\left(Z,\nabla_X\overline{\xi}\right)\left(hY + h^2\varphi Y\right) + g\left(Z,\nabla_Y\overline{\xi}\right)\left(hX + h^2\varphi X\right) \\ & - \overline{\eta}(Z)\left((\nabla_Xh)Y - (\nabla_Yh)X\right) + (\kappa+1)\overline{\eta}(Z)\left((\nabla_X\varphi)Y - (\nabla_Y\varphi)X\right) \\ & + \mu\left[\left(g\left(X,\nabla_Y\overline{\xi}\right) - g\left(Y,\nabla_X\overline{\xi}\right)\right)\varphi hZ - \overline{\eta}(Y)(\nabla_X\varphi h)Z + \overline{\eta}(X)(\nabla_Y\varphi h)Z\right] \\ & + (\nu-2)\left[\left(g\left(Y,\nabla_X\overline{\xi}\right) - g\left(X,\nabla_Y\overline{\xi}\right)\right)hZ + \overline{\eta}(Y)(\nabla_Xh)Z - \overline{\eta}(X)(\nabla_Yh)Z\right]. \end{aligned} \tag{37}$$

If we take $X, Y, Z \in \Gamma(\mathcal{D})$ then from (37), using identities (35), (27) and (8), we get (36). □

Lemma 4.8. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition. Then for each $X, Y, Z \in \Gamma(TM)$ we have*

$$\begin{aligned} R_{XY}\varphi Z - \varphi R_{XY}Z = {} & [\kappa(\overline{\eta}(Y)g(\varphi X,Z) - \overline{\eta}(X)g(\varphi Y,Z)) \\ & + \mu(\overline{\eta}(Y)g(\varphi hX,Z) - \overline{\eta}(X)g(\varphi hY,Z)) - \nu(\overline{\eta}(Y)g(hX,Z) - \overline{\eta}(X)g(hY,Z))]\overline{\xi} \\ & s[-g(Z,\varphi Y + hY)\left(\varphi^2X + \varphi hX\right) + g(Z,\varphi X + hX)\left(\varphi^2Y + \varphi hY\right) \\ & + g\left(Z,\varphi^2X + \varphi hX\right)(\varphi Y + hY) - g\left(Z,\varphi^2Y + \varphi hY\right)(\varphi X + hX)] \\ & - \overline{\eta}(Z)[\kappa(\overline{\eta}(Y)\varphi X - \overline{\eta}(X)\varphi Y) + \mu(\overline{\eta}(Y)\varphi hX - \overline{\eta}(X)\varphi hY) \\ & - \nu(\overline{\eta}(Y)hX - \overline{\eta}(X)hY)]. \end{aligned}$$

Proof. We proceed fixing a point $x \in M$ and local vector fields X, Y, Z such that ∇X, ∇Y and ∇Z vanish at x. Applying several times (27), using (8) and the symmetry of $\nabla\varphi^2$, we get in x

$$\begin{aligned} \nabla_X((\nabla_Y\varphi)Z) - \nabla_Y((\nabla_X\varphi)Z) & \\ = {} & [g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X, Z) + g((\nabla_Xh)Y - (\nabla_Yh)X, Z)]\overline{\xi}s \\ & \times [g(Z,\varphi X + hX)\left(\varphi^2Y + \varphi hY\right) - g(Z,\varphi Y + hY)\left(\varphi^2X + \varphi hX\right) \\ & + g\left(Z,\varphi^2X + \varphi hX\right)(\varphi Y + hY) - g\left(Z,\varphi^2Y + \varphi hY\right)(\varphi X + hX)] \\ & - \overline{\eta}(Z)[((\nabla_X\varphi)Y - (\nabla_Y\varphi)X) + ((\nabla_Xh)Y - (\nabla_Yh)X)]. \end{aligned}$$

From the last identity, using $R_{XY}\varphi Z - \varphi R_{XY}Z = \nabla_X(\nabla_Y\varphi)Z - \nabla_Y(\nabla_X\varphi)Z$ and (28), we get the claimed identity. □

Remark 4.5. *In particular, from Lemma 9 it follows that for a Kenmotsu f-manifold $(M,\varphi,\xi_i,\eta_j,g)$ the following formula holds, for all $X, Y, Z \in \Gamma(TM)$,*

$$\begin{aligned} R_{XY}\varphi Z - \varphi R_{XY}Z = {} & (\overline{\eta}(X)g(\varphi Y,Z) - \overline{\eta}(Y)g(\varphi X,Z)) \\ & s[-g(Z,\varphi Y)\varphi^2X + g(Z,\varphi X)\varphi^2Y + g\left(Z,\varphi^2X\right)\varphi Y - g\left(Z,\varphi^2Y\right)\varphi X] \\ & - \overline{\eta}(Z)[(\overline{\eta}(Y)\varphi X - \overline{\eta}(X)\varphi Y)]. \end{aligned}$$

Theorem 2. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa < -1$. Then for each $X_+, Y_+, Z_+ \in \Gamma(\mathcal{D}_+)$, $X_-, Y_-, Z_- \in \Gamma(\mathcal{D}_-)$, we have*

$$\begin{aligned} R_{X_-Y_-}Z_+ = {} & s(\kappa+1)[g(\varphi Y_-,Z_+)\varphi X_- - g(\varphi X_-,Z_+)\varphi Y_-] \\ & + s\lambda[g(\varphi X_-,Z_+)Y_- - g(\varphi Y_-,Z_+)X_-], \end{aligned}$$

$$\begin{aligned} R_{X_+Y_+}Z_+ = {} & s[g(X_+,Z_+)Y_+ - g(Y_+,Z_+)X_+] \\ & + s\lambda[g(Y_+,Z_+)\varphi X_+ - g(X_+,Z_+)\varphi Y_+], \end{aligned} \tag{38}$$

$$\begin{aligned} R_{X_+Y_+}Z_- = {} & s\lambda[g(Z_-,\varphi Y_+)X_+ - g(Z_-,\varphi X_+)Y_+] \\ & + s(\kappa+1)[g(Z_-,\varphi Y_+)\varphi X_+ - g(Z_-,\varphi X_+)\varphi Y_+], \end{aligned}$$

$$\begin{aligned} R_{X_+Y_-}Z_- = {} & -sg(Y_-,Z_-)X_+ + s(\kappa+1)g(\varphi X_+,Z_-)\varphi Y_- \\ & + s\lambda[g(Y_-,Z_-)\varphi X_+ - g(\varphi X_+,Z_-)Y_-], \end{aligned} \tag{39}$$

$$\begin{aligned} R_{X_+Y_-}Z_+ = {} & sg(X_+,Z_+)Y_- - s(\kappa+1)g(\varphi Y_-,Z_+)\varphi X_+ \\ & + s\lambda[g(X_+,Z_+)\varphi Y_- - g(\varphi Y_-,Z_+)X_+], \end{aligned} \tag{40}$$

$$\begin{aligned} R_{X_-Y_-}Z_- = {} & s[g(X_-,Z_-)Y_- - g(Y_-,Z_-)X_-] \\ & - s\lambda[g(Y_-,Z_-)\varphi X_- - g(X_-,Z_-)\varphi Y_-]. \end{aligned} \tag{41}$$

Proof. First of all, for any $X_+, Y_+, Z_+ \in \mathcal{D}_+$, applying Lemma 7, we get

$$\lambda R_{X_+Y_+}Z_+ - hR_{X_+Y_+}Z_+ = 2s\lambda^2(g(Z_+,Y_+)\varphi X_+ - g(Z_+,X_+)\varphi Y_+)$$

and by scalar multiplication with $W_- \in \mathcal{D}_-$, one has

$$2\lambda(R_{X_+Y_+}Z_+,W_-) = 2s\lambda^2(g(Z_+,Y_+)g(\varphi X_+,W_-) - g(Z_+,X_+)g(\varphi Y_+,W_-))$$

from which, being $\lambda \neq 0$,

$$(R_{X_+Y_+}Z_+,W_-) = s\lambda(g(Z_+,Y_+)g(\varphi X_+,W_-) - g(Z_+,X_+)g(\varphi Y_+,W_-)). \tag{42}$$

With a similar argument, for any $X_+, W_+ \in \mathcal{D}_+$ and $Y_-, Z_- \in \mathcal{D}_-$, we also obtain

$$(R_{X_+Y_-}Z_-,W_+) = (\kappa+1)s(g(Z_-,\varphi X_+)g(\varphi Y_-,W_+) - g(Z_-,Y_-)g(X_+,W_+)) \tag{43}$$

and, from (42), by symmetries of the tensor field R, for any $X_+, Y_+, W_+ \in \mathcal{D}_+$ and $Z_- \in \mathcal{D}_-$

$$(R_{X_+Y_+}Z_-,W_+) = s\lambda(g(Z_-,\varphi Y_+)g(X_+,W_+) - g(Z_-,\varphi X_+)g(Y_+,W_+)). \tag{44}$$

Next, fixed a local φ-basis $\{e_1,\ldots,e_n,\varphi e_1,\ldots,\varphi e_n,\xi_1,\ldots,\xi_s\}$, with $e_i \in \mathcal{D}_+$ we compute $R_{X_+Y_+}Z_-$. The nullity condition implies $g(R_{X_+Y_+}Z_-,\xi_i) = 0$, while using the first Bianchi identity, (43) and (44), we get

$$\begin{aligned} g(R_{X_+Y_+}Z_-,e_i) &= \lambda s(g(Z_-,\varphi Y_+)g(X_+,e_i) - g(Z_-,\varphi X_+)g(Y_+,e_i)), \\ g(R_{X_+Y_+}Z_-,\varphi e_i) &= (\kappa+1)s(g(\varphi Z_-,X_+)g(Y_+,e_i) - g(\varphi Z_-,Y_+)g(X_+,e_i)), \end{aligned}$$

so that, summing on i, the expression for $R_{X_+Y_+}Z_-$ follows.

The terms $R_{X_-Y_-}Z_+$ and $R_{X_+Y_-}Z_-$ are computed in a similar maner. Now, acting by φ on the formula just proved and using Lemma 10, we get

$$R_{X_+Y_+}\varphi Z_- = s(g(\varphi Y_+,Z_-)X_+ - g(\varphi X_+,Z_-)Y_+) - s\lambda(g(\varphi Y_+,Z_-)\varphi X_+ - g(\varphi X_+,Z_-)\varphi Y_+).$$

Writing this formula for φZ_-, by the compatibility condition, we have the result for $R_{X_+Y_+}Z_+$. Similar computation yields $R_{X_-Y_-}Z_-$. Analogously, using the third formula and Lemma 10 we obtain $R_{X_+Y_-}Z_+$. □

Now we are able to compute sectional curvature.

Theorem 3. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then the sectional curvature K of M is determined by*

$$K(X,\xi_i)=\kappa g(X,X)+\mu g(hX,X)+\nu g(\varphi hX,X)=\begin{cases}\kappa+\mu\lambda & \text{if } X\in\mathcal{D}_+,\\ \kappa-\mu\lambda & \text{if } X\in\mathcal{D}_-,\end{cases} \tag{45}$$

$$K(X,Y)=\begin{cases} s & \text{if } X,Y\in\mathcal{D}_+,\\ s & \text{if } X,Y\in\mathcal{D}_-,\\ -s-s(\kappa+1)(g(X,\varphi Y)) & \text{if } X\in\mathcal{D}_+,\ Y\in\mathcal{D}_-.\end{cases} \tag{46}$$

Proof. Identities (45) follow directly from (5), while identities (46) are consequences of (38), (41) and (39) respectively. □

Corollary 4.1. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then the Ricci operator verifies the following identities*

$$Q=s\left[(-2)\varphi^2+\mu h+(2(n-1)+\nu)(\varphi\circ h)\right]+2n\kappa\overline{\eta}\otimes\overline{\xi}, \tag{47}$$

$$Q\circ\varphi-\varphi\circ Q=2s\left[\mu h\circ\varphi+((n-1)+\nu)h\right]. \tag{48}$$

Proof. Let $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi_1,\dots,\xi_s\}$ be a local φ-basis such that $\{e_1,\dots,e_n\}$ is a basis of $\mathcal{D}_+$ and let $X=X_++X_-\in\mathcal{D}_+\oplus\mathcal{D}_-$. From (38), (39) and (10) we get

$$QX_+=s(-2+\mu\lambda)X_++s(2\lambda(n-1)+\nu)\varphi X_+. \tag{49}$$

On the other hand from (40) and (41) we obtain

$$QX_+=s(-2-\mu\lambda)X_+-s(2\lambda(n-1)+\nu)\varphi X_+. \tag{50}$$

Taking into account (49), (50) and $Q\xi_i=2n\kappa\overline{\xi}$, we get (47). Finally, identity (48) easily follows from (47). □

Corollary 4.2. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then the scalar curvature of (M,g) is constant and verifies the following identity*

$$S=2ns(\kappa(2-n)-2n). \tag{51}$$

Proof. Let $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi_1,\dots,\xi_s\}$ be a local φ-basis such that $\{e_1,\dots,e_n\}$ is a basis of $\mathcal{D}_+$ Then from (38), (39) and (5) we have

$$g(Qe_i,e_i)=ksn+\mu\lambda sn-s(\kappa+1)n^2-sn^2. \tag{52}$$

Furthermore, from (40), (41) and (5) we get

$$g(Q\varphi e_i,\varphi e_i)=ksn-\mu\lambda sn-s(\kappa+1)n^2-sn^2. \tag{53}$$

Then (52), (53) and (21) yield (51). □

5 EXAMPLES

Example 1. *Let R^{2n+s} be $(2n+s)$-dimensional real vector space with standard coordinates $(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_s)$ and*

$$M=\{(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_s)\,|z_i\neq0,\ 1\le i\le s,\ n\in N,\ n\ge1\}$$

be a $(2n+s)$-dimensional manifold. For each $i=1,\dots,n$ and $k=1,\dots,s$

$$X_i=\left(-(z_i+1)\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\frac{\partial}{\partial x_i}+e^{z_i}\frac{\partial}{\partial z_i},$$
$$Y_i=\left(z_i+1\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\frac{\partial}{\partial y_i},$$
$$\xi_i=\frac{\partial}{\partial z_i},$$

is a basis of M.

Then, for each $i,j=1,\dots,n$ and $k=1,\dots,s$ we obtain

$$[X_i,Y_j]=e^{z_i}\left(2z_i+3+2e^{2z_i}\right)\frac{\partial}{\partial y_i},\quad [Y_i,Y_j]=0,$$
$$[X_i,\xi_i]=\left(2z_i+3+2e^{2z_i}\right)\frac{\partial}{\partial x_i}-e^{z_i}\frac{\partial}{\partial z_i},\quad [Y_i,\xi_i]=\left(2z_i+1+2e^{2z_i}\right)\frac{\partial}{\partial y_i},$$
$$[X_i,X_j]=-e^{z_i}\left(2z_i+3-2e^{2z_i}\right)\frac{\partial}{\partial x_j}+e^{z_i}\left(2z_i+3-2e^{2z_i}\right)\frac{\partial}{\partial x_i}.$$

If we take $\eta_i=\frac{\partial}{\partial z_i}$, we get

$$g=\sum_{i=1}^{n}\left(\frac{-1}{(z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}}dx_i^2+\frac{1}{(z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}}dy_i^2\right)+\sum_{j=1}^{s}dz_j^2,$$
$$\varphi\xi_i=0,\quad \varphi\left(\frac{\partial}{\partial x_i}\right)=-\frac{\partial}{\partial y_i},$$
$$\varphi\left(\frac{\partial}{\partial y_i}\right)=\frac{\partial}{\partial x_i}-\frac{e^{z_i}}{2(z_i+1)\pm\sqrt{(2z_i+2)^2+4e^{2z_i}}}\frac{\partial}{\partial z_i}.$$

Then, we have an almost metric f-structure (φ,ξ_j,η_i,g) on M. On the other hand, for each $i=1,\dots,s$ we obtain $d\eta_i=0$. Moreover

$$\Phi_{ii}:=g\left(\frac{\partial}{\partial x_i},\varphi\frac{\partial}{\partial y_i}\right)=-\frac{1}{\left(-(z_i+1)\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\left((z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}\right)},$$

and for each $i,j=1,\dots,s$ $\Phi_{ij}=0$. Then we get

$$\Phi_{ii}:=g\left(\frac{\partial}{\partial x_i},\varphi\frac{\partial}{\partial y_i}\right)=$$
$$=-\frac{1}{\left(-(z_i+1)\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\left((z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}\right)}dx_i\wedge dyi,$$

and

$$d\Phi = 2\sum_{j=1}^{s} dz_j \wedge \left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = 2\overline{\eta} \wedge \Phi.$$

Since the Nijenhuis torsion tensor of this manifold is not equal to zero and in view of this expression we get an almost Kenmotsu f-manifold.

Example 2. *Let R^{2n+s} be $(2n+s)$-dimensional real vector space with standard coordinates $(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_s)$ and*

$$M = \{(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_s)\,|z_i \neq 0,\ 1 \leq i \leq s,\ n \in N,\ n \geq 1\}$$

be a $(2n+s)$-dimensional manifold. For each $i=1,\dots,n$ and $k=1,\dots,s$

$$X_i = \left(-1 \pm \sqrt{1+e^{2z_i}}\right)\frac{\partial}{\partial x_i} + z_i^2\frac{\partial}{\partial z_i},$$
$$Y_i = \left(1 \pm \sqrt{1+e^{2z_i}}\right)\frac{\partial}{\partial y_i},$$
$$\xi_i = \frac{\partial}{\partial z_i},$$

is a basis of M.

Then, for each $i,j=1,\dots,n$ and $k=1,\dots,s$, we obtain

$$[X_i, Y_j] = 2z_i^2 e^{2z_i}\frac{\partial}{\partial y_i}, \quad [Y_i, Y_j] = 0,$$
$$[X_i, \xi_i] = -2e^{2z_i}\frac{\partial}{\partial x_i} - z_i^2\frac{\partial}{\partial z_i}, \quad [Y_i, \xi_i] = \pm 2e^{2z_i}\frac{\partial}{\partial y_i},$$
$$[X_i, X_j] = 2z_j^2 e^{2z_i}\frac{\partial}{\partial x_j} - 2z_i^2 e^{2z_i}\frac{\partial}{\partial x_i}.$$

If we take $\eta_i = \frac{\partial}{\partial z_i}$, we get

$$g = \sum_{i=1}^{n}\left(\frac{-1}{1 \pm \sqrt{1+e^{2z_i}}}dx_i^2 + \frac{1}{1 \pm \sqrt{1+e^{2z_i}}}dy_i^2\right) + \sum_{j=1}^{s} dz_j^2,$$
$$\varphi\xi_i = 0, \quad \varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i},$$
$$\varphi\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i} - \frac{z_i^2}{2 \pm \sqrt{4+4e^{2z_i}}}\frac{\partial}{\partial z_i}.$$

Then, we have a metric f-structure $(\varphi, \xi_j, \eta_i, g)$ on M. On the other hand, for each $i=1,\dots,s$ we obtain $d\eta_i = 0$. Moreover

$$\Phi_{ii} := g\left(\frac{\partial}{\partial x_i}, \varphi\frac{\partial}{\partial y_i}\right) = -\frac{1}{\left(-1 \pm \sqrt{1+e^{2z_i}}\right)\left(1 \pm \sqrt{1+e^{2z_i}}\right)},$$

and for each $i,j=1,\dots,s$ $\Phi_{ij} = 0$. Then we get

$$\Phi_{ii} := g\left(\frac{\partial}{\partial x_i}, \varphi\frac{\partial}{\partial y_i}\right) = -\frac{1}{\left(-1 \pm \sqrt{1+e^{2z_i}}\right)\left(1 \pm \sqrt{1+e^{2z_i}}\right)}dx_i \wedge dyi,$$

and

$$d\Phi = 2\sum_{j=1}^{s} dz_j \wedge \left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = 2\overline{\eta} \wedge \Phi.$$

Since the Nijenhuis torsion tensor of this manifold is equal to 0 and in view of these expressions we get a Kenmotsu f-manifold.

REFERENCES

[1] Blair D.E. *Geometry of manifolds with structural group $U(n) \times O(s)$*. J. Differential Geom. 1970, **4**, 155–167.

[2] Cabrerizo J.L., Fernandez L.M., Fernandez M. *The curvature tensor fields on f-manifolds with complemented frames*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N. S.) 1990, **36**, 151–161.

[3] Falcitelli M., Pastore M. *Almost Kenmotsu f-manifolds*. Balkan J. Geom. Appl. 2007, **12** (1), 32–43.

[4] Goldberg S.I., Yano K. *Globally framed f-manifolds*. Illinois J. Math. 1971, **15**, 456–474.

[5] Goldberg S.I., Yano K. *On normal globally framed f-manifolds*. Tohoku Math. J. (2) 1970, **22**, 362–370.

[6] Öztürk H., Aktan N., Murathan C., Vanlı T.A. *Almost α-cosymplectic f-manifolds*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N. S.) 2014, **60** (1), 211–226. doi:10.2478/aicu-2013-0030.

[7] Terlizzi L. D., Konderak J.J., Pastore A.M. *On the flatness of a class of metric f-manifolds*. Bull. Belgian Math. Soc. Simon Stevin 2003, **10**, 461–474.

[8] Yano K. *On a structure defined by a tensor field f of type $(1,1)$ satisfying $f^3 + f = 0$*. Tensor (N.S) 1963, **14**, 99–109.

[9] Yano K., Kon M. Structures on Manifolds. In: Series in Pure Math, 3. World Sci, 1984.

Received 14.01.2014

Revised 30.10.2014

Балкан Я.С., Актан Н. *Майже Кенмотсу f-многовиди* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 6–21.

В статті розглядаються узагальнення майже Кенмотсу f-многовидів. Отримано основні властивості Ріманової кривизни, секційних кривин і скалярної кривизни для таких типів многовидів. Насамкінець наведено два приклади.

Ключові слова і фрази: f-структура, майже Кенмотсу f-многовиди.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 22–27
doi:10.15330/cmp.7.1.22-27

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.22–27

VASYLYSHYN P.B.[1], SAVKA I.YA.[2], KLYUS I.S.[3]

MULTIPOINT NONLOCAL PROBLEM FOR FACTORIZED EQUATION WITH DEPENDENT COEFFICIENTS IN CONDITIONS

The conditions of correct solvability of multipoint nonlocal problem for factorized PDE with coefficients in conditions, which depend on one real parameter, are established. It is shown that these conditions on the set of full Lebesgue measure of the interval parameters are fulfilled.

Key words and phrases: differential equations, multipoint nonlocal problem, dependent coefficients, small denominators, diophantine approximation, metric estimations.

[1] Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
[2] Institute for Applied Problems of Mechanics and Mathematics, 3b Naukova str., 79060, Lviv, Ukraine
[3] National Aviation University, 1 Komarova str., 03058, Kyiv, Ukraine
E-mail: pbvasylyshyn@ukr.net (Vasylyshyn P.B.), s-i@ukr.net (Savka I.Ya.), isklyus@gmail.com (Klyus I.S.)

Let $\mathbb{T}^p$ denote the p-dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^p$, $T>0$, $Q_p^T = (0,T)\times\mathbb{T}^p$, $\Pi_n = \{\lambda = (\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n : \lambda_i \neq \lambda_j \text{ if } i\neq j\}$, $D_x = (-i\partial/\partial x_1,\dots,-i\partial/\partial x_p)$, $k=(k_1,\dots,k_p)\in\mathbb{Z}^p$, $|k| = |k_1|+\dots+|k_p|$; $B(D_x)$ is differential expression such that

$$\exists\, N_1, N_2 \in \mathbb{R},\ C_1, C_2 > 0: \quad (\forall k\in\mathbb{Z}^p) \quad C_1(1+|k|)^{N_1} \le |B(k)| \le C_2(1+|k|)^{N_2}. \tag{1}$$

We use the following functional spaces: $\mathbf{H}_q = \mathbf{H}_q(\mathbb{T}^p)$, $q\in\mathbb{R}$, is Sobolev space obtained by completing the space of all finite trigonometric polynomials

$$\varphi(x) = \sum_k \varphi_k \exp(ik,x)$$

by the norm

$$\|\varphi;\mathbf{H}_q\| = \left(\sum_{k\in\mathbb{Z}^p}(1+|k|^2)^q|\varphi_k|^2\right)^{1/2}.$$

Let us denote by $\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)$, $n\in\mathbb{Z}_+$, $\theta\in\mathbb{R}$, space of functions

$$u(t,x) = \sum_{k\in\mathbb{Z}^p} u_k(t)\exp(ik,x)$$

such that for any fixed point $t\in[0,T]$ function

$$\partial^j u(t,x)/\partial t^j \equiv \sum_{k\in\mathbb{Z}^p} u_k^{(j)}(t)\exp(ik,x)$$

belong to the space $\mathbf{H}_{q-j\theta}$, $j=0,1,\dots,n$, and it, as an element of this space, is continuous in t on $[0,T]$; the norm in $\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)$ is defined as follows

$$\|u;\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)\|^2 = \sum_{j=0}^n \max_{t\in[0,T]} \|\partial^j u(t,x)/\partial t^j;\mathbf{H}_{q-j\theta}\|^2.$$

УДК 517.95+511.42
2010 *Mathematics Subject Classification:* 35G15, 11K60.

In the domain Q_p^T we consider the following problem:

$$L(\partial/\partial t, D_x)u \equiv \prod_{j=1}^n \left(\frac{\partial}{\partial t} - \lambda_j B(D_x)\right) u(t,x) = 0, \quad (t,x)\in Q_p^T, \tag{2}$$

$$L_j u \equiv \sum_{r=1}^m \mu_r(\tau) \frac{\partial^{j-1} u(t,x)}{\partial t^{j-1}}\Big|_{t=t_r} = \varphi_j(x), \quad x\in\mathbb{T}^p,\ j=1,\dots,n, \tag{3}$$

where $(\lambda_1,\dots,\lambda_n)\in\Pi_n$, the real-valued coefficients $\mu_1,\dots,\mu_m$ depend on parameters τ, $\tau\in I$, where I is an arbitrary fixed segment of the line $\mathbb{R}$, $t_1,\dots,t_m$ are the points of the interval $[0,T]$, and $0=t_1<t_2<\dots<t_{m-1}<t_m=T$.

Solvability of boundary value problems with multipoint nonlocal conditions for parabolic, strictly hyperbolic, typeless and pseudodifferential equations studied in works [1–4, 6–10].

The problem (2), (3) belong to a class of incorrect problems by Hadamard and its solvability related to the problem of small denominators. In the assumption when the coefficients $\mu_1,\dots,\mu_m$ are independent correct solvability of the problem (2), (3) follows from the results of [10, §14]. If the coefficients $\mu_1,\dots,\mu_m$ are dependent, then these results will not be used to proving solvability of the problem (2), (3). It shoud be noted that two-point nonlocal problem for partial differential equation of the n-th order with conditions (3) was investigated in [11] for the case of two-point nonlocal conditions ($m=2$).

In the paper we found that the conditions of correct solvability of the multipoint nonlocal problem (2), (3) in the scale of Sobolev spaces are fulfilled for almost all (with respect to Lebesgue measure in the space $\mathbb{R}$) numbers $\tau\in I$.

The solution u to the problem (2), (3) has the form of a Fourier series

$$u(t,x) = \sum_{k\in\mathbb{Z}^p} u_k(t)\exp(ik,x), \tag{4}$$

where function $u_k(t)$, $k\in\mathbb{Z}^p$, is a solution of multipoint nonlocal problem of ordinary differential equations:

$$L(d/dt,k)u_k(t) = 0, \tag{5}$$

$$L_j u_k(t) = \varphi_{jk}, \qquad j=1,\dots,n. \tag{6}$$

Here, φ_{jk} are Fourier coefficients of the function $\varphi_j(x)$, $j=1,\dots,n$.

For each fixed $k\in\mathbb{Z}^p$ let us construct a solution to problem (5), (6). Since the $(\lambda_1,\dots,\lambda_n)\in\Pi_n$ and coefficients $B(k)$ satisfy the condition (1), the equation (5) for each $k\in\mathbb{Z}^p$ has the fundamental system of solutions $\{e^{\lambda_1 B(k)t},\dots,e^{\lambda_n B(k)t}\}$. Then the general solution of the equation (5) has the form

$$u_k(t) = c_{1k}\exp(\lambda_1 B(k)t) + \dots + c_{nk}\exp(\lambda_n B(k)t),$$

where constants $c_{1k},\dots,c_{nk}$ are determined from conditions (6) with the help of the system of linear equations

$$\begin{aligned} c_{1k}\lambda_1^{j-1}\sum_{r=1}^m \mu_r(\tau)\exp(\lambda_1 B(k)t_r) + c_{2k}\lambda_2^{j-1}\sum_{r=1}^m \mu_r(\tau)\exp(\lambda_2 B(k)t_r) \\ + c_{nk}\lambda_n^{j-1}\sum_{r=1}^m \mu_r(\tau)\exp(\lambda_n B(k)t_r) = \varphi_{jk}B^{1-j}(k), \qquad j=1,\dots,n. \end{aligned} \tag{7}$$

Determinant of the system (7) is factorized and represented by the formula

$$\Delta_k(\tau) = W(\lambda)\prod_{s=1}^{n}\Phi_{sk}(\tau),$$

where

$$\Phi_{sk}(\tau) = \sum_{r=1}^{m}\mu_r(\tau)\exp(\lambda_s B(k)t_r),$$

$W(\lambda) = \prod\limits_{1\le\alpha<\beta\le n}(\lambda_\alpha - \lambda_\beta)$ is Vandermonde determinant constructed from different numbers $\lambda_1,\dots,\lambda_n$, hense $W(\lambda)\neq 0$.

If $\prod\limits_{s=1}^{n}\Phi_{sk}(\tau)\neq 0$, then

$$c_{sk} = \frac{1}{W(\lambda)\Phi_{sk}(\tau)}\sum_{j=1}^{n}(-1)^{s+j}W_{js}(\lambda)B^{1-j}(k)\varphi_{jk}, \quad s=1,\dots,n.$$

Here we denote by $W_{js}(\lambda)$ the Vandermonde-type determinant obtained from the determinant $W(\lambda)$ by crossing out j-row and s-column.

Thus, the solution to the problem (5), (6) under the condition $\prod\limits_{s=1}^{n}\Phi_{sk}(\tau)\neq 0$ is unique and has the following form

$$u_k(t) = \sum_{s,j=1}^{n}\frac{(-1)^{s+j}W_{js}(\lambda)B^{1-j}(k)}{W(\lambda)\Phi_{sk}(\tau)}\varphi_{jk}\exp(\lambda_s B(k)t), \qquad k\in\mathbb{Z}^p. \tag{8}$$

Conditions for uniqueness of the solution u of the problem (2), (3) follows from the theorem on uniqueness of Fourier expansion of a periodic function and from conditions of uniqueness of the solution $u_k(t)$ of the problem (5), (6) for each $k\in\mathbb{Z}^p$.

Theorem 1. *For uniqueness (at fixed parameter τ) of the solution of the problem* (2), (3) *in the space $\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)$ it is necessary and sufficient that for all $k\in\mathbb{Z}^p$*

$$\Phi_{1k}(\tau)\Phi_{2k}(\tau)\cdot\ldots\cdot\Phi_{nk}(\tau)\neq 0. \tag{9}$$

If the condition (9) holds, then the formal solution of the problem (2), (3) is represented by the formula

$$u(t,x) = \sum_{k\in\mathbb{Z}^p}\sum_{s,j=1}^{n}\frac{(-1)^{s+j}W_{js}(\lambda)B^{1-j}(k)}{W(\lambda)\Phi_{sk}(\tau)}\varphi_{jk}\exp\big(\lambda_s B(k)t + (ik,x)\big). \tag{10}$$

Expressions $\Phi_{1k}(\tau),\Phi_{2k}(\tau),\dots,\Phi_{nk}(\tau)$ influence the convergence of the series (10), which determines the norm of the solution of the problem (2), (3) in the space $\mathbf{C}^n_\theta([0,T];\mathbf{H}_q)$. This is explained by the fact that the denominators $\Phi_{1k}(\tau),\Phi_{2k}(\tau),\dots,\Phi_{nk}(\tau)$, $k\in\mathbb{Z}^p$, although non vanishing by the condition above, can arbitrarily rapidly approach to zero for infinite set of vectors $k\in\mathbb{Z}^p$. Therefore, the existence of the solution u of the problem related to the so called problem of small denominators.

To solve this problem we use the metric approach [5] to estimations from below of small denominators.

At first, we formulate the corresponding theorem from the work [12].

Theorem 2. *Let*

$$F(\tau,z) = f_1(\tau)z_1 + \ldots + f_m(\tau)z_m,$$

where $z=(z_1,\dots,z_m)\in\mathbb{C}^m$, and $\{f_1,\dots,f_m\}\subset C^m(I;\mathbb{R})$. If the Wronskian $W[f_1,\dots,f_m]$ of the functions $f_1,\dots,f_m$ is not equal to zero on the interval $I\subset\mathbb{R}$, then for all $z\in\mathbb{C}^m\setminus\{0\}$ and an arbitrary $\varepsilon\in(0,C_1|z|/2)$, the following evaluation is valid

$$\operatorname{meas}\{\tau\in I : |F(\tau,z)|<\varepsilon\}\le C_2\sqrt[m-1]{\varepsilon/|z|},$$

where $|z| = |z_1|+\ldots+|z_m|$, positive constants C_1 and C_2 are defined by formulas

$$C_1 = \frac{1}{m}\min_{\tau\in I}|W[f_1,\dots,f_m](\tau)|\Big(\prod_{j=1}^{m}\|f_j\|_{C^{(m-1)}(I;\mathbb{R})}\sum_{j=1}^{m}\|f_j\|^{-1}_{C^{(m-1)}(I;\mathbb{R})}\Big)^{-1},$$

$$C_2 = 4(\sqrt{2}+1)(m-1)C_1^{m/(1-m)}\big(\operatorname{meas} I\max_{1\le j,q\le m}\|f_j^{(q)}\|_{C(I;\mathbb{R})} + C_1\big).$$

Theorem 3. *If $\mu_r\in C^m(I)$, $r=1,\dots,m$, and Wronskian $W[\mu_1,\dots,\mu_m]$ of functions $\mu_1,\dots,\mu_m$ is not equal to zero on the interval I, then for almost all (with respect to Lebesgue measure in the space $\mathbb{R}$) numbers $\tau\in I$ evaluations*

$$|\Phi_{sk}(\tau)|\ge|k|^{-\gamma}\max\big(1,\exp(\mathrm{Re}\lambda_s B(k)T)\big), \quad s=1,\dots,n, \tag{11}$$

are satisfied for all (except perhaps a finite number) vectors $k\in\mathbb{Z}^p$ for $\gamma>p(m-1)$.

Proof. For fixed s we introduce the sets

$$B^s_k = \{\tau\in I\colon |\Phi_{sk}(\tau)|<\varepsilon_k\}, \qquad k\in\mathbb{Z}^p,$$

and the set B^s of such points $\tau\in I$, for which infinite times on $\mathbb{Z}^p$ the estimate is true

$$|\Phi_{sk}(\tau)|<\varepsilon_k = \frac{C_1|k|^{-\gamma}}{2}\max\big(1,\exp(\mathrm{Re}\lambda_s B(k)T)\big), \quad \delta>0.$$

If $z(s,k) = \big(e^{\lambda_s B(k)t_1},\dots,e^{\lambda_s B(k)t_m}\big)$, $f_j(\tau)=\mu_j(\tau)$ for $j=1,\dots,m$, then from Theorem 2 follow the equalities:

$$F(\tau,z(s,k)) = \Phi_{sk}(\tau), \qquad W[f_1,\dots,f_m] = W[\mu_1,\dots,\mu_m].$$

Since

$$|z(s,k)| = 1 + |e^{\lambda_s B(k)t_2}| + \ldots + |e^{\lambda_s B(k)t_{m-1}}| + +|e^{\lambda_s B(k)T}| \ge \max\big(1,\exp(\mathrm{Re}\lambda_s B(k)T)\big)$$

for all $k\in\mathbb{Z}^p\setminus\{0\}$ and the inequalities are fulfilled

$$0<\varepsilon_k<\frac{C_1}{2}\max\big(1,\exp(\mathrm{Re}\lambda_s B(k)T)\big) < \frac{C_1}{2}|z(s,k)|$$

then for each $k\neq 0$ by conditions of Theorem 2 we have the following estimation for the measure B^s_k

$$\operatorname{meas} B^s_k \le C_2\sqrt[m-1]{\varepsilon_k/|z(s,k)|}\le C_3|k|^{-\gamma/(m-1)}, \qquad C_3 = C_2\Big(\frac{C_1}{2}\Big)^{1/(m-1)}.$$

For selected $\gamma > p(m-1)$ series $\sum\limits_{k\in\mathbb{Z}^p\setminus\{0\}}$ meas B_k^s is majorized by the convergent series $C_3 \sum\limits_{k\in\mathbb{Z}^p} |k|^{-\delta/(p-1)}$. Then from the Borel–Cantelli lemma follows that Lebesgue measure of the set of points τ from I, which contained into the infinite number of sets B_k^s, is equal to zero for fixed s. Thus, meas $B^s = 0$ for all $s = 1, \dots, n$.

Therefore, when $\gamma > p(m-1)$ for almost all (with respect to Lebesgue measure in $\mathbb{R}$) numbers $\tau \in I$ inequality $|\Phi_{sk}(\tau)| \geq \varepsilon_k$, $s = 1, \dots, n$, is satisfied for all (except for a finite number of) vectors k. The theorem is proved. □

Theorem 4. *Let the condition* (9) *is valid,* $\min\limits_{\tau\in I} |W[\mu_1, \dots, \mu_m](\tau)| > 0$, $\mu_r \in C^m(I)$, $r = 1, \dots, m$, *and* $\varphi_j \in \mathbf{H}_{q+N_1(1-j)+\gamma}$, *where* $\gamma > p(m-1)$, $j = 1, \dots, n$. *Then for almost all (with respect to Lebesgue measure in the space* $\mathbb{R}$*) numbers* $\tau \in I$ *there exists a unique solution of the problem* (2), (3) *in the space* $\mathbf{C}^n_{N_2}([0,T]; \mathbf{H}_q)$, *which is represented by a series* (10) *and continuously depends on the functions* φ_j, $j = 1, \dots, n$.

Proof. Taking into account, that

$$\left|\frac{W_{js}(\lambda)}{W(\lambda)}\right| \leq M_1, \ M_1 = M_1(\lambda),$$

on the basis of formula (10) and estimations (1), (11) we obtain the inequality

$$\|u; \mathbf{C}^n_{N_2}([0,T]; \mathbf{H}_q)\|^2 \leq M_2 \sum_{j=1}^{n} \|\varphi_j; \mathbf{H}_{q+\gamma+N_1(1-j)}\|^2,$$

where $M_2 = 2^{N_2 n+\gamma} n^3 (n+1) M_1^2 C_2^{2n} |\lambda|^{2n}$, $|\lambda| = \max\limits_{1\leq s\leq n} |\lambda_s|$. The proof of the theorem is complete. □

REFERENCES

[1] Gorodetskii V.V., Drin' Ya.M. *Nonlocal multipoint with respect to time problem for evolution pseudo-differential equations in spaces of periodic functions*. Bukovyna Math. J. 2014, **2** (1), 26–42. (in Ukrainian)

[2] Il'kiv V.S. *Multipoint nonlocal boundary value problem for a system of nonhomogeneous partial differential equations with time variable continuous coefficients*. Math. Bull. Shevchenko Sci. Soc. 2004, 1, 47–58. (in Ukrainian)

[3] Il'kiv V.S. *The smoothness of solutions of the problems with nonlocal multi-point conditions for strictly hyperbolic equations*. Carpathian Math. Publ. 2009, **1** (1), 47–58. (in Ukrainian)

[4] Il'kiv V.S. *Time-Variable Extension of the Solution of a Nonlocal Multipoint Problem for Partial Differential Equations with Constant Coefficients*. J. Math. Sci. 2001, **107** (1), 3615–3619. doi:10.1023/A:1011954509838 (translation of Mat. Metodi Fiz.-Mekh. Polya 1998, **41** (4), 78–82. (in Ukrainian))

[5] Il'kiv V.S., Ptashnyk B.I. *Problems for partial differential equations with nonlocal conditions. Metric approach to the problem of small denominators*. Ukrainian Math. J. 2006, **58** (12), 1847-1875. doi:10.1007/s11253-006-0172-8 (translation of Ukrain. Mat. Zh. 2006, **58** (12), 1624–1650. (in Ukrainian))

[6] Il'kiv V.S., Maherovska T.V. *Boundary-value problem with non-local multipoint conditions for hyperbolic equation*. Math. Methodi Fiz.-Mekh. Polya 2007, **50** (3), 66–81. (in Ukrainian)

[7] Il'kiv V.S., Maherovska T.V. *Nonlocal multi-point problem for high order strong hyperbolic equation*. Math. Bull. Shevchenko Sci. Soc. 2007, 4, 107–115. (in Ukrainian)

[8] Martynyuk O.V. *The multipoint problem for one class of differential-operational equations*. Dopov. Nats. Akad. Nauk Ukr. 2011, 10, 19–24. (in Ukrainian)

[9] Martynyuk O.V., Gorodetskii V.V. *Multipoint problem for a class of evolution equations*. Differ. Equ. 2013, **49** (8), 975–985. doi:10.1134/S0012266113080065 (translation of Differ. Uravn. 2013, **49** (8), 1005–1015. (in Russian))

[10] Ptashnyk B.Yo., Il'kiv V.S., Kmit' I.Ya., Polishchuk V.M. Nonlocal boundary value problems for partial differential equations. Naukova Dumka, Kyiv, 2002. (in Ukrainian)

[11] Savka I.Ya. *Nonlocal boundary value problem for partial differential equations, constant coefficients of which lie on smooth curves*. Math. Methodi Fiz.-Mekh. Polya 2009, **52** (4), 18–33. (in Ukrainian)

[12] Savka I.Ya. *Nonlocal problem with dependent coefficients in conditions for the second-order equation in time variable*. Carpathian Math. Publ. 2010, **2** (2), 101–110. (in Ukrainian)

Received 17.03.2015

Revised 11.06.2015

Василишин П.Б., Савка І.Я., Клюс І.С. *Багатоточкова нелокальна задача для факторизованого рівняння з залежними коефіцієнтами в умовах* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 22–27.

Встановлено умови коректної розв'язності нелокальної багатоточкової задачі для факторизованого рівняння з коефіцієнтами в умовах, що залежать від одного дійсного параметра. Показано, що ці умови виконуються на множині повної міри Лебега відрізка параметрів.

Ключові слова і фрази: диференціальні рівняння, багатоточкова нелокальна задача, залежні коефіцієнти, малі знаменники, діофантові наближення, метричні оцінки.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 28–37
doi:10.15330/cmp.7.1.28-37

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.28–37

ВЕЛЬГАЧ А.В.

НЕПЕРЕРВНО-ДИФЕРЕНЦІЙОВНІ РОЗВ'ЯЗКИ ОДНІЄЇ ГРАНИЧНОЇ ЗАДАЧІ ДЛЯ СИСТЕМ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНО-РІЗНИЦЕВИХ РІВНЯНЬ НЕЙТРАЛЬНОГО ТИПУ ТА ЇХ ВЛАСТИВОСТІ

Встановлено достатні умови існування неперервно-диференційовних і обмежених при $t \in \mathbb{R}^+$ розв'язків однієї граничної задачі для систем лінійних диференціально-різницевих рівнянь нейтрального типу зі скінченною кількістю постійних відхилень аргументу, запропоновано метод їх побудови та досліджено асимптотичні властивості таких розв'язків.

Ключові слова і фрази: неперервно-диференційовний обмежений розв'язок, гранична задача, система лінійних диференціально-різницевих рівнянь нейтрального типу.

Volodymyr Gnatiuk Ternopil National Pedagogical University, 2 Kryvonosa str., 46027, Ternopil, Ukraine
E-mail: velgandr@ukr.net

У даній статті розглядається система рівнянь вигляду

$$x'(t+1) = Ax'(t) + \sum_{m=1}^{k} A_m(t)x(t+\alpha_m) + \sum_{m=1}^{k} B_m(t)x'(t+\beta_m) + F(t), \tag{1}$$

де A — стала $(n \times n)$-матриця, $A_m(t)$, $B_m(t)$, $m = 1,\ldots,k$ — неперервні при $t \in \mathbb{R}^+ = [0,+\infty)$ $(n \times n)$-матриці, $\alpha_m > 0$, $\beta_m > 0$, $m = 1,..,k$, $F(t)$ — неперервна при $t \in \mathbb{R}^+$ вектор-функція розмірності n. Зауважимо, що системи вигляду

$$x'(t+1) = Ax'(t) + F(t, x(t), x(f(t)), x'(g(t))), \tag{2}$$

були предметом розгляду багатьох математиків [1, 2]. При цьому, як правило, вивчалися задачі, які характерні для звичайних диференціальних рівнянь — існування і єдиність розв'язків задачі Коші, основної початкової задачі, різного роду крайових задач. Але при дослідженні таких рівнянь дуже часто виникає необхідність в дослідженні задач, які враховують їх специфіку. Одна із таких задач полягає в дослідженні питання про існування неперервно-диференційовних при $t \in \mathbb{R}^+$ розв'язків систем (2), які задовольняють умову

$$\lim_{t\to+\infty} [x(t+1) - Ax(t)] = 0. \tag{3}$$

Зокрема, в [3, 4] у випадку коли $A = E$ та [5] при $det A \neq 0$ вивчалася структура множини неперервно-диференційовних при $t \in \mathbb{R}^+$ розв'язків граничної задачі (2), (3).

В даній роботі досліджується питання існування неперервно-диференційовних і обмежених при $t \in \mathbb{R}^+$ розв'язків задачі (1), (3) у випадку, коли виконуються наступні умови:

УДК 517.9
2010 *Mathematics Subject Classification:* 37H10.

1) $\left|\int_t^{+\infty} F(\tau)d\tau\right| \leq M$, $|F(t)| \leq M$, де M — деяка додатна стала, $t \in \mathbb{R}^+$,
$\left|\int_t^{+\infty} |A_m(\tau)|d\tau\right| \leq a_m$, $|A_m(t)| \leq a_m$, $\left|\int_t^{+\infty} |B_m(\tau)|d\tau\right| \leq b_m$, $|B_m(t)| \leq b_m$,
$m = 1,..,k$, $t \in \mathbb{R}^+$;

2) $|A^{-1}| < 1$, $\Delta = \dfrac{|A^{-1}|}{1-|A^{-1}|}\left(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m\right) < 1$.

Для розв'язання задачі (1), (3) достатньо, очевидно, довести, що система інтегральних рівнянь

$$x(t+1) = Ax(t) - \int_t^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)x(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)x'(\tau+\beta_m) + F(\tau)\Big)d\tau \tag{4}$$

має неперервно-диференційовний при $t \in \mathbb{R}^+$ розв'язок.

Покажемо, що система (4) має розв'язок у вигляді ряду

$$\overline{x}(t) = \sum_{i=0}^{\infty} x_i(t), \tag{5}$$

де $x_i(t)$, $i = 0, 1, \ldots,$ — деякі неперервно-диференційовні при $t \in \mathbb{R}^+$ вектор-функції.

Дійсно, підставляючи ряд (5) в (4), отримуємо

$$\sum_{i=0}^{\infty} x_i(t+1) = A\sum_{i=0}^{\infty} x_i(t) - \int_t^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)\sum_{i=0}^{\infty} x_i(\tau+\alpha_m)$$
$$+ \sum_{m=1}^{k} B_m(\tau)\sum_{i=0}^{\infty} x_i'(\tau+\beta_m) + F(\tau)\Big)d\tau.$$

Звідси безпосередньо випливає, що якщо вектор-функції $x_i(t)$, $i = 0, 1, \ldots,$ є розв'язками послідовності систем рівнянь

$$x_0(t+1) = Ax_0(t) - \int_t^{+\infty} F(\tau)d\tau, \tag{6}$$

$$x_i(t+1) = Ax_i(t) - \int_t^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)x_{i-1}(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)x_{i-1}'(\tau+\beta_m)\Big)d\tau, \quad i = 1, 2, \ldots, \tag{7}$$

то ряд (5) є формальним розв'язком системи рівнянь (4).

Система рівнянь (6) має формальний розв'язок у вигляді ряду

$$x_0(t) = \sum_{j=0}^{\infty} A^{-(j+1)} \int_{t+j}^{+\infty} F(\tau)d\tau, \tag{8}$$

який внаслідок умов 1), 2) рівномірно збігається при $t \in \mathbb{R}^+$ разом із своєю похідною і виконуються умови

$$|x_0(t)| \le \widetilde{M}, \quad |x_0'(t)| \le \widetilde{M}, \tag{9}$$

де $\widetilde{M} = \frac{|A^{-1}|}{1-|A^{-1}|}M$. Оскільки при $i = 1, 2, \ldots$ ряди

$$x_i(t) = \sum_{j=0}^{\infty} A^{-(j+1)} \int\limits_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau) x_{i-1}(\tau + \alpha_m) + \sum_{m=1}^{k} B_m(\tau) x_{i-1}'(\tau + \beta_m) \Big) d\tau, \tag{10}$$

є формальними розв'язками відповідних систем рівнянь (7) (в цьому можна переконатися безпосередньою підстановкою (10) в (7)), то залишається показати, що вони рівномірно збігаються при $t \in \mathbb{R}^+$ разом із своїми похідними і виконуються оцінки:

$$|x_i(t)| \le \widetilde{M}\Delta^i, \quad |x_i'(t)| \le \widetilde{M}\Delta^i, \quad i = 1, 2, \ldots. \tag{11}$$

Справді, при $i = 1$ маємо

$$|x_1(t)| \le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int\limits_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)||x_0(\tau+\alpha_m)| + \sum_{m=1}^{k} |B_m(\tau)||x_0'(\tau+\beta_m)| \Big) d\tau \Big|$$

$$\le \widetilde{M} \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big(\sum_{m=1}^{k} \Big| \int\limits_{t+j}^{+\infty} |A_m(\tau)| d\tau \Big| + \sum_{m=1}^{k} \Big| \int\limits_{t+j}^{+\infty} |B_m(\tau)| d\tau \Big| \Big)$$

$$\le \widetilde{M} \frac{|A^{-1}|}{1-|A^{-1}|} \Big(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m \Big) \le \widetilde{M}\Delta,$$

$$|x_1'(t)| \le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big(\sum_{m=1}^{k} |A_m(t+j)||x_0(t+j+\alpha_m)| + \sum_{m=1}^{k} |B_m(t+j)||x_0'(t+j+\beta_m)| \Big)$$

$$\le \widetilde{M} \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m \Big) \le \widetilde{M}\Delta$$

і, отже, оцінки (11) мають місце. Розмірковуючи по індукції, припустимо, що оцінка (11) доведена уже для деякого $i \ge 1$, і покажемо, що вона зберігається для $i+1$. Дійсно, на підставі (10), (11) і умов 1), 2) отримуємо

$$|x_{i+1}(t)| \le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int\limits_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)||x_i(\tau+\alpha_m)| + \sum_{m=1}^{k} |B_m(\tau)||x_i'(\tau+\beta_m)| \Big) d\tau \Big|$$

$$\le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \widetilde{M}\Delta^i \Big(\sum_{m=1}^{k} \Big| \int\limits_{t+j}^{+\infty} |A_m(\tau)| d\tau + \sum_{m=1}^{k} \Big| \int\limits_{t+j}^{+\infty} |B_m(\tau)| d\tau \Big) \Big|$$

$$\le \widetilde{M}\Delta^i \frac{|A^{-1}|}{1-|A^{-1}|} \Big(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m \Big) \le \widetilde{M}\Delta^{i+1},$$

$$|x_{i+1}'(t)| \le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big(\sum_{m=1}^{k} |A_m(t+j)||x_i(t+j+\alpha_m)| + \sum_{m=1}^{k} |B_m(t+j)||x_i'(t+j+\beta_m)| \Big)$$

$$\le \widetilde{M}\Delta^i \frac{|A^{-1}|}{1-|A^{-1}|} \Big(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m \Big) \le \widetilde{M}\Delta^{i+1}.$$

Отже, системи рівнянь (7), $i = 0, 1, \ldots$, мають неперервно-диференційовні при $t \in \mathbb{R}^+$ розв'язки $x_i(t)$, $i = 0, 1, \ldots$, у вигляді рядів (10), $i = 0, 1, \ldots$, які рівномірно збігаються при всіх $t \in \mathbb{R}^+$, і задовольняють умови (11), $i = 0, 1, \ldots$. Звідси і умови 2) безпосередньо випливає, що ряд (5) (разом із своєю першою похідною) рівномірно збігається при всіх $t \in \mathbb{R}^+$, його сума $\overline{x}(t)$ є неперервно-диференційовним розв'язком системи рівнянь (4) і задовольняє умови

$$|\overline{x}(t)| \le \frac{\widetilde{M}}{1-\Delta}, \quad |\overline{x}'(t)| \le \frac{\widetilde{M}}{1-\Delta}.$$

Покажемо тепер, що побудований у вигляді ряду (5) розв'язок $\overline{x}(t)$ системи рівнянь (4) є єдиним при виконанні умов 1), 2). Дійсно, припустимо, що існує ще один неперервно-диференційовний обмежений при $t \in \mathbb{R}^+$ розв'язок $y(t)$ такий, що $y(t) \not\equiv \overline{x}(t)$. Тоді із тотожностей

$$\overline{x}(t+1) = A\overline{x}(t) - \int\limits_{t}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)\overline{x}(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)\overline{x}'(\tau+\beta_m) + F(\tau) \Big) d\tau,$$

$$y(t+1) = Ay(t) - \int\limits_{t}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)y(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)y'(\tau+\beta_m) + F(\tau) \Big) d\tau$$

і умов 1), 2) отримуємо

$$|\overline{x}(t) - y(t)| \le |A^{-1}||\overline{x}(t+1) - y(t+1)| + |A^{-1}| \Big| \int\limits_{t}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)||\overline{x}(\tau+\alpha_m) - y(\tau+\alpha_m)|$$

$$+ \sum_{m=1}^{k} |B_m(\tau)||\overline{x}'(\tau+\alpha_m) - y'(\tau+\beta_m)| \Big) d\tau \Big|$$

$$\le |A^{-1}| \, \|\overline{x}(t) - y(t)\| + |A^{-1}| \big(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m \big) \|\overline{x}(t) - y(t)\|$$

$$= (|A^{-1}| + \Delta(1 - |A^{-1}|)) \|\overline{x}(t) - y(t)\| = \Delta' \|\overline{x}(t) - y(t)\|,$$

$$|\overline{x}'(t) - y'(t)| \le |A^{-1}||\overline{x}'(t+1) - y'(t+1)| + |A^{-1}| \Big| \Big(\sum_{m=1}^{k} |A_m(t)||\overline{x}(t+\alpha_m) - y(t+\alpha_m)|$$

$$+ \sum_{m=1}^{k} |B_m(t)||\overline{x}'(t+\alpha_m) - y'(t+\beta_m)| \Big) d\tau \Big|$$

$$\le |A^{-1}||\overline{x}(t) - y(t)\| + |A^{-1}| \big(\sum_{m=1}^{k} a_m + \sum_{m=1}^{k} b_m \big) \|\overline{x}(t) - y(t)\|$$

$$= (|A^{-1}| + \Delta(1 - |A^{-1}|)) \|\overline{x}(t) - y(t)\| = \Delta' \|\overline{x}(t) - y(t)\|,$$

де $\|\overline{x}(t) - y(t)\| = max\Big\{ \sup\limits_{t\in\mathbb{R}^+} |\overline{x}(t) - y(t)|, \sup\limits_{t\in\mathbb{R}^+} |\overline{x}'(t) - y'(t)| \Big\}$, $0 < \Delta' < 1$. Звідси випливає $\|\overline{x}(t) - y(t)\| \le \Delta' \|\overline{x}(t) - y(t)\|$, що є можливим лише у випадку, коли $\overline{x} \equiv y$. Отже, отримане протиріччя показує, що побудований вище неперервно-диференційовний обмежений при $t \in \mathbb{R}^+$ розв'язок $\overline{x}(t)$ у вигляді ряду (5) є єдиним при виконанні умов 1), 2).

Підсумовуючи наведені вище результати, приходимо до наступної теореми.

Теорема 1. *Нехай виконуються умови 1), 2). Тоді система рівнянь* (4) *має єдиний неперервно-диференційовний обмежений при* $t \in \mathbb{R}^+$ *розв'язок* $\widetilde{x}(t)$ *у вигляді ряду* (5), *в якому вектор-функції* $x_i(t)$, $i = 0, 1, \ldots$, *визначаються формулами* (10), $i = 0, 1, \ldots$.

Розглянемо тепер систему рівнянь вигляду (1) у випадку, коли виконуються умови:

3) Ряди $\widetilde{F}(t) = \sum\limits_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int\limits_{t+j}^{+\infty} |F(\tau)| d\tau \Big|$, $\widetilde{F}'(t) = \sum\limits_{j=0}^{\infty} |A^{-1}|^{j+1} |F(t+j)|$ рівномірно збігаються при всіх $t \in \mathbb{R}^+$ і $\widetilde{F}(t) \le P$, $\widetilde{F}'(t) \le P$, $P > 0$;

4) $|A_m(t)| \le a_m(t)$, $|B_m(t)| \le b_m(t)$, $m = 1, \ldots, k$, $\sum\limits_{m=1}^{k} a_m(t) = a(t)$, $\sum\limits_{m=1}^{k} b_m(t) = b(t)$, де $a_m(t)$, $b_m(t)$, $m = 1, \ldots, k$, — деякі неперервні при $t \in \mathbb{R}^+$, невід'ємні функції такі, що

$$\sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} (a(\tau) + b(\tau)) d\tau \Big| \le \theta < 1,$$
$$\sum_{j=0}^{\infty} |A^{-1}|^{j+1} \big(a(t+j) + b(t+j)\big) \le \theta < 1.$$

Як і раніше будемо досліджувати питання про існування неперервно-диференційовних при $t \in \mathbb{R}^+$ розв'язків, що задовольняють умову (3). Для цього, очевидно, достатньо вивчити це питання для системи інтегральних рівнянь (4). Має місце наступна теорема.

Теорема 2. *Нехай виконуються умови 3), 4). Тоді система рівнянь* (4) *має неперервно-диференційовний обмежений при* $t \in \mathbb{R}^+$ *розв'язок* $\widetilde{x}(t)$.

Доведення. Покажемо, що при виконанні умов 3), 4) система рівнянь (4) має неперервно-диференційовний обмежений при $t \in \mathbb{R}^+$ розв'язок $\widetilde{x}(t)$ у вигляді ряду

$$\widetilde{x}(t) = \sum_{i=0}^{\infty} \widetilde{x}_i(t), \tag{12}$$

де $\widetilde{x}_i(t)$, $i = 0, 1, \ldots$, — деякі неперервно-диференційовні обмежені при $t \in \mathbb{R}^+$ вектор-функції. Дійсно, підставляючи ряд (12) у (4), одержимо

$$\sum_{i=0}^{\infty} \widetilde{x}_i(t+1) = A \sum_{i=0}^{\infty} \widetilde{x}_i(t) - \int_t^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau) \sum_{i=0}^{\infty} \widetilde{x}_i(\tau + \alpha_m) + \sum_{m=1}^{k} B_m(\tau) \sum_{i=0}^{\infty} \widetilde{x}'_i(\tau + \beta_m) + F(\tau) \Big) d\tau,$$

звідки приходимо до висновку, що якщо вектор-функції $\widetilde{x}_i(t)$, $i = 0, 1, \ldots$, є розв'язками послідовності систем рівнянь

$$\widetilde{x}_0(t+1) = A\widetilde{x}_0(t) - \int_t^{+\infty} F(\tau) d\tau, \tag{13}$$

$$\widetilde{x}_i(t+1) = A\widetilde{x}_i(t) - \int_t^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau) \widetilde{x}_{i-1}(\tau + \alpha_m) + \sum_{m=1}^{k} B_m(\tau) \widetilde{x}'_{i-1}(\tau + \beta_m) \Big) d\tau, \quad i = 1, 2, \ldots, \tag{14}$$

то ряд (12) є формальним розв'язком системи (4). Згідно умови 3) ряди

$$\widetilde{x}_0(t) = \sum_{j=0}^{\infty} A^{-(j+1)} \int_{t+j}^{+\infty} F(\tau) d\tau, \qquad \widetilde{x}'_0(t) = -\sum_{j=0}^{\infty} A^{-(j+1)} F(t+j) \tag{15}$$

рівномірно збігаються при всіх $t \in \mathbb{R}^+$, вектор-функція $\widetilde{x}_0(t)$ задовольняє систему рівнянь (13) (в цьому можна переконатися безпосередньою підстановкою (15) в (13)) і умови

$$|\widetilde{x}_0(t)| \le P, \tag{16}$$
$$|\widetilde{x}'_0(t)| \le P. \tag{17}$$

Розглядаючи послідовно системи рівнянь (14), $i = 1, 2, \ldots$, можна переконатися, що ряди

$$\widetilde{x}_i(t) = \sum_{j=0}^{\infty} A^{-(j+1)} \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau) \widetilde{x}_{i-1}(\tau + \alpha_m) + \sum_{m=1}^{k} B_m(\tau) \widetilde{x}'_{i-1}(\tau + \beta_m) \Big) d\tau, \quad i = 1, 2, \ldots, \tag{18}$$

є формальними розв'язками відповідних систем рівнянь (14), $i = 1, 2, \ldots$. Доведемо, що ці ряди рівномірно збігаються при $t \in \mathbb{R}^+$ до деяких вектор-функцій $\widetilde{x}_i(t)$, $i = 1, 2, \ldots$, які є неперервно-диференційовними і задовольняють умови

$$|\widetilde{x}_i(t)| \le P\theta^i, \quad i = 1, 2, \ldots, \tag{19}$$
$$|\widetilde{x}'_i(t)| \le P\theta^i, \quad i = 1, 2, \ldots. \tag{20}$$

Дійсно, на підставі умови 4), (18), (16) і (17) отримуємо

$$|\widetilde{x}_1(t)| \le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)| |\widetilde{x}_0(\tau + \alpha_m)| + \sum_{m=1}^{k} |B_m(\tau)| |\widetilde{x}'_0(\tau + \beta_m)| \Big) d\tau \Big| \le P \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} (a(\tau) + b(\tau)) d\tau \Big| \le P\theta,$$

тобто в цьому випадку оцінка (19) має місце. Оскільки згідно 4) ряд

$$\sum_{j=0}^{\infty} A^{-(j+1)} \Big(\sum_{m=1}^{k} A_m(t+j) \widetilde{x}_0(t+j+\alpha_m) + \sum_{m=1}^{k} B_m(t+j) \widetilde{x}'_0(t+j+\beta_m) \Big)$$

рівномірно збігається при всіх $t \in \mathbb{R}^+$, то вектор-функція $\widetilde{x}_1(t)$ є неперервно-диференційовною при $t \in \mathbb{R}^+$ і має місце оцінка (20).

Розмірковуючи по індукції, припустимо, що співвідношення (19), (20) доведені уже для деякого $i \geq 1$, і доведемо, що вони зберігаються при переході від i до $i+1$. Справді, внаслідок (18), (19), (20) і умови 4) отримуємо

$$|\widetilde{x}_{i+1}(t)| \leq \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)||\widetilde{x}_i(\tau+\alpha_m)| + \sum_{m=1}^{k} |B_m(\tau)||\widetilde{x}_i'(\tau+\beta_m)| \Big) d\tau \Big|$$

$$\leq P\theta^i \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} (a(\tau)+b(\tau)) d\tau \Big| \leq P\theta^{i+1}.$$

Вектор-функція $\widetilde{x}_{i+1}(t)$ є неперервно-диференційовною при $t \in \mathbb{R}^+$ і виконується оцінка (20). Це випливає із 4), (19), (20) і рівномірної збіжності ряду

$$\sum_{j=0}^{\infty} A^{-(j+1)} \Big(\sum_{m=1}^{k} A_m(t+j)\widetilde{x}_i(t+j+\alpha_m) + \sum_{m=1}^{k} B_m(t+j)\widetilde{x}_i'(t+j+\beta_m) \Big).$$

Отже, оцінки (19), (20) мають місце при всіх $i \geq 1$. Звідси безпосередньо випливає, що ряд (12) рівномірно збігається при $t \in \mathbb{R}^+$ до деякої неперервно-диференційовної вектор-функції $\widetilde{x}(t)$, яка є розв'язком системи рівнянь (4) і задовольняє умову

$$|\widetilde{x}(t)| \leq \frac{P}{1-\theta}. \tag{21}$$

Теорема 2 доведена. □

Таким чином, на підставі теореми 2 система рівнянь (4) має неперервно-диференційовний обмежений при $t \in \mathbb{R}^+$ розв'язок $\widetilde{x}(t)$ у вигляді ряду (12). Більше цього, далі ми покажемо, що при деяких додаткових умовах система рівнянь (4) має нескінченно багато неперервно-диференційовних обмежених при $t \in \mathbb{R}^+$ розв'язків $x(t) = x(t, \omega(t))$, де $\omega(t)$ — деяка 1-періодична вектор-функція, які задовольняють умову

$$\lim_{t\to+\infty} [x(t) - \widetilde{x}(t)] = 0. \tag{22}$$

Виконаємо в (4) взаємно-однозначну заміну змінних

$$x(t) = y(t) + \widetilde{x}(t), \tag{23}$$

де $\widetilde{x}(t)$ — розв'язок системи (4) у вигляді ряду (12). У результаті отримаємо систему рівнянь

$$y(t+1) = Ay(t) - \int_{t}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)y(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)y'(\tau+\beta_m) \Big) d\tau, \tag{24}$$

відносно якої будемо припускати виконаня умови 4) і умови

5) $|A|<1$.

Має місце наступна теорема.

Теорема 3. *Нехай виконуються умови 4), 5). Тоді система рівнянь (24) має сім'ю неперервно-диференційовних обмежених при $t \in \mathbb{R}^+$ розв'язків $y(t) = y(t, \omega(t))$ у вигляді ряду*

$$y(t) = \sum_{i=0}^{\infty} y_i(t), \tag{25}$$

де $y_i(t)$, $i = 0, 1, \ldots,$ — деякі неперервно-диференційовні обмежені при $t \in \mathbb{R}^+$ вектор-функції, які задовольняють умову

$$\lim_{t\to+\infty} y(t) = 0. \tag{26}$$

Доведення. Ряд (25) є формальним розв'язком системи рівнянь (24), тобто виконується співвідношення

$$\sum_{i=0}^{\infty} y_i(t+1) = A\sum_{i=0}^{\infty} y_i(t) - \int_{t}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau) \sum_{i=0}^{\infty} y_i(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau) \sum_{i=0}^{\infty} y_i'(\tau+\beta_m) \Big) d\tau,$$

у випадку, коли вектор-функції $y_i(t)$, $i = 0, 1, \ldots,$ є розв'язками послідовності систем рівнянь

$$y_0(t+1) = Ay_0(t), \tag{27}$$

$$y_i(t+1) = Ay_i(t) - \int_{t}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)y_{i-1}(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)y_{i-1}'(\tau+\beta_m) \Big) d\tau, \quad i = 1, 2, \ldots. \tag{28}$$

Система рівнянь (27) має сім'ю неперервно-диференційовних при $t \in \mathbb{R}^+$ розв'язків вигляду

$$y_0(t) = A^{[t]}\varphi(t-[t]), \tag{29}$$

де $[t]$ — ціла частина t, $\varphi(\tau)$ — довільна неперервно-диференційовна при $\tau \in [0,1)$ вектор-функція, що задовольняє умови

$$\varphi(1-0) = A\varphi(0), \quad \varphi'(1-0) = A\varphi'(0).$$

Легко переконатися, що якщо $y_0(t)$ є один із розв'язків, що визначаються формулою (29), то мають місце оцінки

$$|y_0(t)| \leq \widetilde{P}|A|^t, \tag{30}$$

$$|y_0'(t)| \leq \widetilde{P}|A|^t, \tag{31}$$

де $\widetilde{P}$ — деяка додатна стала.

Розглядаючи послідовно системи рівнянь (28), $i = 1, 2, \ldots,$ можна переконатися, що ряди

$$y_i(t) = \sum_{j=0}^{\infty} A^{-(j+1)} \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} A_m(\tau)y_{i-1}(\tau+\alpha_m) + \sum_{m=1}^{k} B_m(\tau)y_{i-1}'(\tau+\beta_m) \Big) d\tau, \tag{32}$$

є формальними розв'язками відповідних систем рівнянь (28), $i = 1, 2, \ldots$. Покажемо тепер, що ряди (32), $i = 1, 2, \ldots$, рівномірно збігаються при всіх $t \in \mathbb{R}^+$ до деяких вектор-функцій $y_i(t)$, $i = 1, 2, \ldots$, які є неперервно-диференційовними при $t \in \mathbb{R}^+$ і задовольняють умови

$$|y_i(t)| \le \widetilde{P}\theta^i |A|^t, \quad i = 1, 2, \ldots, \tag{33}$$

$$|y_i'(t)| \le \widetilde{P}\theta^i |A|^t, \quad i = 1, 2, \ldots. \tag{34}$$

Справді, приймаючи до уваги (32), (30), (31) і умови 4), 5), отримуємо

$$\begin{aligned}
|y_1(t)| &\le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)||y_0(\tau+\alpha_m)| + \sum_{m=1}^{k} |B_m(\tau)||y_0'(\tau+\beta_m)| \Big) d\tau \Big| \\
&\le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} a_m(\tau)\widetilde{P}|A|^{\tau+\alpha_m} + \sum_{m=1}^{k} b_m(\tau)\widetilde{P}|A|^{\tau+\beta_m} \Big) d\tau \Big| \\
&\le \widetilde{P} \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} a_m(\tau)|A|^{\tau-t+\alpha_m} + \sum_{m=1}^{k} b_m(\tau)|A|^{\tau-t+\beta_m} \Big) d\tau \Big| |A|^t \\
&\le \widetilde{P} \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} (a(\tau)+b(\tau)) d\tau \Big| |A|^t \le \widetilde{P}\theta |A|^t.
\end{aligned}$$

Отже, оцінка (33) має місце. Диференціюючи (32), отримуємо ряд

$$y_1'(t) = -\sum_{j=0}^{\infty} A^{-(j+1)} \Big(\sum_{m=1}^{k} A_m(t+j) y_0(t+j+\alpha_m) + \sum_{m=1}^{k} B_m(t+j) y_0'(t+j+\beta_m) \Big),$$

який на підставі умов 4), 5) і співвідношень (30), (31) рівномірно збігається при $t \in \mathbb{R}^+$ і його сума $y_1'(t)$ задовольняє умову $|y_1'(t)| \le \widetilde{P}\theta |A|^t$.

Аналогічно можна довести, що ряди (32), $i = 1, 2, \ldots, r$, рівномірно збігаються при $t \in \mathbb{R}^+$ до деяких неперервно-диференційовних вектор-функцій $y_i(t)$, $i = 1, 2, \ldots, r$, що задовольняють умови (33), (34), $i = 1, 2, \ldots, r$. На підставі (32), (33), (34) і умов 4), 5) отримуємо

$$\begin{aligned}
|y_{r+1}(t)| &\le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} |A_m(\tau)||y_r(\tau+\alpha_m)| + \sum_{m=1}^{k} |B_m(\tau)||y_r'(\tau+\beta_m)| \Big) d\tau \Big| \\
&\le \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} a_m(\tau)\widetilde{P}\theta^r|A|^{\tau+\alpha_m} + \sum_{m=1}^{k} b_m(\tau)\widetilde{P}\theta^r|A|^{\tau+\beta_m} \Big) d\tau \Big| \\
&\le \widetilde{P}\theta^r \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} \Big(\sum_{m=1}^{k} a_m(\tau)|A|^{\tau-t+\alpha_m} + \sum_{m=1}^{k} b_m(\tau)|A|^{\tau-t+\beta_m} \Big) d\tau \Big| |A|^t \\
&\le \widetilde{P}\theta^r \sum_{j=0}^{\infty} |A^{-1}|^{j+1} \Big| \int_{t+j}^{+\infty} (a(\tau)+b(\tau)) d\tau \Big| |A|^t \le \widetilde{P}\theta^{r+1} |A|^t.
\end{aligned}$$

Отже, оцінка (33) виконується при всіх $i \ge 1$. Внаслідок (33), (34) і умов 4), 5) ряд

$$y_{r+1}'(t) = -\sum_{j=0}^{\infty} A^{-(j+1)} \Big(\sum_{m=1}^{k} A_m(t+j) y_r(t+j+\alpha_m) + \sum_{m=1}^{k} B_m(t+j) y_r'(t+j+\beta_m) \Big),$$

рівномірно збігається при $t \in \mathbb{R}^+$ і виконується умова $|y_{r+1}'(t)| \le \widetilde{P}\theta^{r+1}|A|^t$.

Тим самим доведено, що ряди (32), $i = 1, 2, \ldots$, рівномірно збігаються при всіх $t \in \mathbb{R}^+$ до деяких неперервно-диференційовних вектор-функцій $y_i(t)$, $i = 1, 2, \ldots$, які задовольняють умови (33), (34), $i = 1, 2, \ldots$. Звідси безпосередньо випливає, що ряд (25) і ряд

$$y'(t) = \sum_{i=0}^{\infty} y_i'(t)$$

рівномірно збігаються при $t \in \mathbb{R}^+$ і виконуються співвідношення

$$|y(t)| \le \frac{\widetilde{P}}{1-\theta}|A|^t, \quad |y'(t)| \le \frac{\widetilde{P}}{1-\theta}|A|^t.$$

Приймаючи до уваги останні співвідношення і умову 5), отримуємо, що побудовані розв'язки задовольняють умову

$$\lim_{t\to+\infty} y(t) = 0.$$

Теорема 3 доведена. □

REFERENCES

[1] Bellman R., Cooke K.L. Differential-Difference Equations. Academic Press, New York, 1963.

[2] Hale J. Theory of Functional Differential Equations. Springer, New York, 1977.

[3] Pelyukh G.P. *Asymptotic properties of solutions to systems of nonlinear functional-differential equations.* Differ. Equ. 2003, **39** (1), 46–50. doi:10.1023/A:1025163807158

[4] Pelyukh G.P. *On the structure of the set of solutions of one class of systems of nonlinear functional differential equations of neutral type.* Nonlinear Oscil. 2002, **5** (1), 53–59. doi:10.1023/A:1014652927323 (translation of Neliniini Koliv. 2002, **5** (1), 58–56. (in Ukrainian))

[5] Vel'hach A.V. *Asymptotic properties of solutions of systems of nonlinear functional differential equations of neutral type.* Nonlinear Oscil. 2009, **12** (1), 19–26. doi:10.1007/s11072-009-0056-6 (translation of Neliniini Koliv. 2009, **12** (1), 20–26. (in Ukrainian))

Надійшло 04.03.2014

Vel'hach A.V. *Continuously differentiable solutions of one boundary value problem for systems of linear difference differential equations of neutral type and their properties.* Carpathian Math. Publ. 2015, **7** (1), 28–37.

Conditions of the existence of continuously differentiable bounded for $t \in \mathbb{R}^+$ solutions of one boundary value problem for systems of linear and nonlinear difference differential equations of neutral type have been obtained. The method of their construction has been developed and the asymptotic properties of these solutions are investigated.

Key words and phrases: continuously differentiable solutions, boundary value problem, system of linear difference differential equations of neutral type.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 38–48
doi:10.15330/cmp.7.1.38-48

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.38–48

GERASIMENKO V.I.

NEW APPROACH TO DERIVATION OF QUANTUM KINETIC EQUATIONS WITH INITIAL CORRELATIONS

We propose a new approach to the derivation of kinetic equations from dynamics of large particle quantum systems, involving correlations of particle states at initial time. The developed approach is based on the description of the evolution within the framework of marginal observables in scaling limits. As a result the quantum Vlasov-type kinetic equation with initial correlations is constructed and the statement relating to the property of a propagation of initial correlations is proved in a mean field limit.

Key words and phrases: marginal observable, kinetic equation with initial correlations.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01601, Kyiv, Ukraine
E-mail: gerasym@imath.kiev.ua

INTRODUCTION

As it is well known the collective behavior of large particle quantum systems can be effectively described within the framework of a one-particle (marginal) density operator governed by the kinetic equation [1–4]. In this paper we consider the problem of the rigorous description of the kinetic evolution in the presence of initial correlations of quantum particles. Such initial states are typical for the condensed states of quantum gases [5–8] in contrast to the gaseous state. For example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition specified by correlations of the condensed state [5]. One more example is the influence of initial correlations on ultrafast relaxation processes in plasmas [9], [10].

The conventional approach to the rigorous derivation of the quantum kinetic equations is based on the consideration of an asymptotic behavior of a solution of the quantum BBGKY hierarchy for marginal density operators constructed within the framework of the theory of perturbations in case of initial states specified by a one-particle (marginal) density operator without correlations [11–14], i.e. such that satisfy a chaos condition. This method of the derivation of quantum kinetic equations can not be extended on case of initial states specified by initial correlations.

In the paper for the rigorous derivation of the quantum kinetic equations in the presence of initial correlations we develop a new approach based on the description of the evolution of large particle quantum systems within the framework of marginal observables governed by the dual quantum BBGKY hierarchy [15]. In article [16] a rigorous formalism of the description of the kinetic evolution of observables of quantum particles in a mean field scaling limit was developed. In this case the limit dynamics is described by the set of recurrence evolution

УДК 517.9+531.19+530.145
2010 *Mathematics Subject Classification:* 74A25.

equations, namely by the dual quantum Vlasov hierarchy. In this paper, using established relationships of initial states specified by initial correlations and constructed solution of the dual quantum Vlasov hierarchy for the limit marginal observables, we derive the quantum Vlasov-type kinetic equation with initial correlations. The statement relating to the property of a propagation of initial correlations is also proved.

1 PRELIMINARY FACTS

We consider a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics in the space $\mathbb{R}^3$. We will use units where $h = 2\pi\hbar = 1$ is a Planck constant, and $m = 1$ is the mass of particles.

Let the space $\mathcal{H}$ be a one-particle Hilbert space, then the n-particle space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ is a tensor product of n Hilbert spaces $\mathcal{H}$. We adopt the usual convention that $\mathcal{H}^{\otimes 0} = \mathbb{C}$. The Fock space over the Hilbert space $\mathcal{H}$ we denote by $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty}\mathcal{H}_n$.

Let $\mathfrak{L}^1(\mathcal{H}_n)$ be the space of trace class operators $f_n \equiv f_n(1,\ldots,n) \in \mathfrak{L}^1(\mathcal{H}_n)$ that satisfy the symmetry condition: $f_n(1,\ldots,n) = f_n(i_1,\ldots,i_n)$ for arbitrary $(i_1,\ldots,i_n) \in (1,\ldots,n)$, and equipped with the norm: $\|f_n\|_{\mathfrak{L}^1(\mathcal{H}_n)} = \mathrm{Tr}_{1,\ldots,n}|f_n(1,\ldots,n)|$, where $\mathrm{Tr}_{1,\ldots,n}$ are partial traces over $1,\ldots,n$ particles. We denote by $\mathfrak{L}^1_0(\mathcal{H}_n)$ the everywhere dense set of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

We shall consider initial states of a quantum many-particle system specified by the one-particle (marginal) density operator $F_1^{0,\varepsilon} \in \mathfrak{L}^1(\mathcal{H})$ in the presence of correlations, i.e. initial states specified by the following sequence of marginal (s-particle) density operators

$$F^{(c)} = \Big(I, F_1^{0,\varepsilon}(1), g_2^{\varepsilon}(1,2)\prod_{i=1}^{2}F_1^{0,\varepsilon}(i),\ldots, g_n^{\varepsilon}(1,\ldots,n)\prod_{i=1}^{n}F_1^{0,\varepsilon}(i),\ldots\Big), \tag{1}$$

where I is an identity operator, the operators $g_n^{\varepsilon}(1,\ldots,n) \equiv g_n^{\varepsilon} \in \mathfrak{L}^1_0(\mathcal{H}_n)$, $n \geq 2$, are specified the initial correlations and the parameter $\varepsilon > 0$ is a mean field scaling parameter [17].

Traditionally correlations of quantum many-particle systems are described within the framework of marginal (s-particle) correlation operators which are introduced by means of the cluster expansions of the marginal density operators

$$F_s^{0,\varepsilon}(1,\ldots,s) = \sum_{\mathrm{P}\,:\,(1,\ldots,s)=\bigcup_i X_i}\ \prod_{X_i\subset \mathrm{P}} G_{|X_i|}^{0,\varepsilon}(X_i), \quad s \geq 1, \tag{2}$$

where $\sum_{\mathrm{P}:(1,\ldots,s)=\bigcup_i X_i}$ is the sum over all partitions P of the set $(1,\ldots,s)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset (1,\ldots,s)$. Hereupon solutions of cluster expansions (2)

$$G_s^{0,\varepsilon}(1,\ldots,s) = \sum_{\mathrm{P}\,:\,(1,\ldots,s)=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset \mathrm{P}} F_{|X_i|}^{0,\varepsilon}(X_i), \quad s \geq 1, \tag{3}$$

are interpreted as the operators that describe correlations. Hence in the case of initial data (1) sequence (3) of marginal correlation operators has the form

$$G^{(c)} = \Big(I, F_1^{0,\varepsilon}(1), \tilde{g}_2^{\varepsilon}(1,2)\prod_{i=1}^{2}F_1^{0,\varepsilon}(i),\ldots, \tilde{g}_n^{\varepsilon}(1,\ldots,n)\prod_{i=1}^{n}F_1^{0,\varepsilon}(i),\ldots\Big), \tag{4}$$

where the operators $\tilde{g}_n^{\varepsilon}(1,\ldots,n) \equiv \tilde{g}_n^{\varepsilon} \in \mathfrak{L}^1_0(\mathcal{H}_n)$, $n \geq 2$, specified the initial correlations are determined by the expansions

$$\tilde{g}_s^{\varepsilon} = \sum_{\mathrm{P}\,:\,Y=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset \mathrm{P}} g_{|X_i|}^{\varepsilon}, \quad s \geq 2. \tag{5}$$

We remark that in case of initial data satisfying a chaos condition [2] sequence (3) of marginal correlation operators has the form

$$G^0 = \big(I, G_1^{0,\varepsilon}(1), 0, \ldots, 0, \ldots\big), \tag{6}$$

and concequently sequence (2) of marginal density operators takes the form

$$F^0 = \big(I, F_1^{0,\varepsilon}(1), \prod_{i=1}^{2} F_1^{0,\varepsilon}(i), \ldots, \prod_{i=1}^{n} F_1^{0,\varepsilon}(i), \ldots\big). \tag{7}$$

Such assumption about initial states, i.e. (7) (or (6)), is intrinsic for the kinetic description of a gas [1]. On the other hand, initial states (1) (or (4)) are typical for the condensed states of quantum gases, for example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition with the correlations which characterize the condensed state [5].

We note that the evolution of large particle quantum systems can be described not only within the framework of marginal density operators governed by the quantum BBGKY hierarchy [2] but also in terms of marginal observables governed by the dual quantum BBGKY hierarchy [15].

Let a sequence $g = (g_0, g_1, \ldots, g_n, \ldots)$ be an infinite sequence of self-adjoint bounded operators g_n defined on the Fock space $\mathcal{F}_{\mathcal{H}}$. An operator g_n defined on the n-particle Hilbert space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ will be also denoted by the symbol $g_n(1, \ldots, n)$. Let the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$ be the space of sequences $g = (g_0, g_1, \ldots, g_n, \ldots)$ of bounded operators g_n defined on the Hilbert space $\mathcal{H}_n$ that satisfy symmetry condition: $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$, for arbitrary $(i_1, \ldots, i_n) \in (1, \ldots, n)$, equipped with the operator norm $\|.\|_{\mathfrak{L}(\mathcal{H}_n)}$. We will also consider a more general space $\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ with the norm $\|g\|_{\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})} \doteq \max_{n\geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)}$, where $0 < \gamma < 1$. We denote by $\mathfrak{L}_{\gamma,0}(\mathcal{F}_{\mathcal{H}}) \subset \mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ the everywhere dense set in the space $\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

In terms of observables the evolution of quantum many-particle systems is described by the sequence $B(t) = (B_0, B_1(t,1), \ldots, B_s(t,1,\ldots,s), \ldots)$ of marginal observables (or s-particle observables) $B_s(t,1,\ldots,s)$, $s \geq 1$, determined by the following expansions [15]:

$$B_s(t,Y) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n = 1}^{s} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X)\, B_{s-n}^{0,\varepsilon}(Y \setminus X), \quad s \geq 1, \tag{8}$$

where $B(0) = (B_0, B_1^{0,\varepsilon}(1), \ldots, B_s^{0,\varepsilon}(1,\ldots,s), \ldots) \in \mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ is a sequence of initial marginal observables, and the generating operator $\mathfrak{A}_{1+n}(t)$ of expansion (8) is the $(1+n)th$-order cumulant of groups of operators (10) defined by the expansion

$$\mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) \doteq \sum_{\mathrm{P}:\,(\{Y\setminus X\}, X) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} \mathcal{G}_{|\theta(X_i)|}(t, \theta(X_i)), \tag{9}$$

where we hold abridged notations: $Y \equiv (1,\ldots,s)$, $X \equiv (j_1,\ldots,j_n) \subset Y$, and $\{Y \setminus X\}$ is the set, consisting of a single element $Y \setminus X = (1,\ldots,s) \setminus (j_1,\ldots,j_n)$, thus, the set $\{Y \setminus X\}$ is a connected subset of the set Y, the symbol $\sum_{\mathrm{P}}$ means the sum over all partitions P of the set $(\{Y \setminus X\}, j_1, \ldots, j_n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset (\{Y \setminus X\}, X)$, and $\theta(\cdot)$ is the declusterization mapping defined as follows: $\theta(\{Y \setminus X\}, X) = Y$. In expansion (9) for $g_n \in \mathfrak{L}(\mathcal{H}_n)$ the one-parameter mapping $\mathcal{G}_n(t)$ is defined by the formula

$$\mathbb{R}^1 \ni t \mapsto \mathcal{G}_n(t) g_n \doteq e^{itH_n} g_n e^{-itH_n}, \tag{10}$$

where the Hamilton operator H_n of a system of n particles is a self-adjoint operator with the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$ has the structure

$$H_n = \sum_{i=1}^{n} K(i) + \varepsilon \sum_{i_1 < i_2 = 1}^{n} \Phi(i_1, i_2), \tag{11}$$

and $K(i)$ is the operator of a kinetic energy of the i particle, $\Phi(i_1, i_2)$ is the operator of a two-body interaction potential and $\varepsilon > 0$ is a scaling parameter [17]. The operator $K(i)$ acts on functions ψ_n, that belong to the subspace $L_0^2(\mathbb{R}^{3n}) \subset \mathcal{D}(H_n) \subset L^2(\mathbb{R}^{3n})$ of infinitely differentiable functions with compact supports, according to the formula: $K(i)\psi_n = -\frac{1}{2}\Delta_{q_i}\psi_n$. Correspondingly, we have: $\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n$, and we assume that the function $\Phi(q_{i_1}, q_{i_2})$ is symmetric with respect to permutations of its arguments, translation-invariant and bounded function.

On the space $\mathfrak{L}(\mathcal{H}_n)$ one-parameter mapping (10) is an isometric $*$-weak continuous group of operators. The infinitesimal generator $\mathcal{N}_n$ of this group of operators is a closed operator for the $*$-weak topology, and on its domain of the definition $\mathcal{D}(\mathcal{N}_n) \subset \mathfrak{L}(\mathcal{H}_n)$ it is defined in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_n)$ by the operator

$$\mathrm{w}^*\!-\lim_{t\to 0} \frac{1}{t}\big(\mathcal{G}_n(t) g_n - g_n\big) = -i\,(g_n H_n - H_n g_n) \doteq \mathcal{N}_n g_n, \tag{12}$$

where H_n is the Hamiltonian (11) and the operator $\mathcal{N}_n g_n$ defined on the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$ has the structure

$$\mathcal{N}_n = \sum_{j=1}^{n} \mathcal{N}(j) + \varepsilon \sum_{j_1 < j_2 = 1}^{n} \mathcal{N}_{\mathrm{int}}(j_1, j_2),$$

where

$$\mathcal{N}(j) g_n \doteq -i\,(g_n K(j) - K(j) g_n), \tag{13}$$

$$\mathcal{N}_{\mathrm{int}}(j_1, j_2) g_n \doteq -i\,(g_n \Phi(j_1, j_2) - \Phi(j_1, j_2) g_n). \tag{14}$$

Therefore on the space $\mathfrak{L}(\mathcal{H}_n)$ a unique solution of the Heisenberg equation for observables of a n-particle system is determined by group (10).

The simplest examples of marginal observables (8) are given by the expansions:

$$B_1(t,1) = \mathfrak{A}_1(t,1) B_1^{0,\varepsilon}(1),$$

$$B_2(t,1,2) = \mathfrak{A}_1(t,\{1,2\}) B_2^{0,\varepsilon}(1,2) + \mathfrak{A}_2(t,1,2)(B_1^{0,\varepsilon}(1) + B_1^{0,\varepsilon}(2)),$$

where the corresponding order cumulants (9) of groups of operators (10) are given by the formulas

$$\mathfrak{A}_1(t,\{1,2\}) = \mathcal{G}_s(t,1,2),$$

$$\mathfrak{A}_2(t,1,2) = \mathcal{G}_s(t,1,2) - \mathcal{G}_1(t,1)\mathcal{G}_1(t,2).$$

If $\gamma < e^{-1}$, for the sequence of operators (8) the following estimate is true: $\|B(t)\|_{\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})} \leq e^2(1-\gamma e)^{-1} \|B(0)\|_{\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})}$.

A sequence of marginal observables (8) is the non-perturbative solution of recurrence evolution equations known as the dual quantum BBGKY hierarchy [15].

We note that in case of initial states specified by sequences (23) the average values (mean values) of marginal observables (8) are determined by the following positive continuous linear functional

$$(B(t), F^{(c)}) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{Tr}_{1,\dots,n} B_n(t, 1, \dots, n) \, g_n^{\varepsilon}(1, \dots, n) \prod_{i=1}^{n} F_1^{0,\varepsilon}(i). \tag{15}$$

For operators $B(t) \in \mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ and $F_1^{0,\varepsilon} \in \mathfrak{L}^1(\mathcal{H})$, functional (24) exists under the condition that $\|F_1^{0,\varepsilon}\|_{\mathfrak{L}^1(\mathcal{H})} < \gamma$.

2 The description of the kinetic evolution within the framework of marginal observables

In scaling limits the kinetic evolution of many-particle systems can be described within the framework of observables. We consider this problem on an example of the mean field asymptotic behavior of non-perturbative solution (8) of the dual quantum BBGKY hierarchy for marginal observables.

A mean field asymptotic behavior of marginal observables (8) is described by the following proposition [16].

Let for $B_n^{0,\varepsilon} \in \mathfrak{L}(\mathcal{H}_n)$, in the sense of the $*$-weak convergence on the space $\mathfrak{L}(\mathcal{H}_n)$ it holds

$$\mathrm{w}^*-\lim_{\varepsilon \to 0} (\varepsilon^{-n} B_n^{0,\varepsilon} - b_n^0) = 0, \quad n \geq 1,$$

then for arbitrary finite time interval there exists mean field scaling limit of marginal observables (8)

$$\mathrm{w}^*-\lim_{\varepsilon \to 0} (\varepsilon^{-s} B_s(t) - b_s(t)) = 0, \quad s \geq 1, \tag{16}$$

that are determined by the following expansions:

$$\begin{aligned} b_s(t, Y) = &\sum_{n=0}^{s-1} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \sum_{i_1 \neq j_1 = 1}^{s} \mathcal{N}_{\mathrm{int}}(i_1, j_1) \\ &\times \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \dots \prod_{l_n \in Y \setminus (j_1, \dots, j_{n-1})} \mathcal{G}_1(t_{n-1} - t_n, l_n) \\ &\times \sum_{\substack{i_n \neq j_n = 1, \\ i_n, j_n \neq (j_1, \dots, j_{n-1})}}^{s} \mathcal{N}_{\mathrm{int}}(i_n, j_n) \prod_{l_{n+1} \in Y \setminus (j_1, \dots, j_n)} \mathcal{G}_1(t_n, l_{n+1}) b_{s-n}^0(Y \setminus (j_1, \dots, j_n)), \end{aligned} \tag{17}$$

where the operator $\mathcal{N}_{\mathrm{int}}(i_1, j_2)$ is defined on operators $g_n \in \mathfrak{L}(\mathcal{H}_n)$ by formula (14).

The proof of this statement is based on formulas for cumulants of asymptotically perturbed groups of operators (10).

Indeed, for arbitrary finite time interval the asymptotically perturbed group of operators (10) has the following scaling limit in the sense of the $*$-weak convergence on the space $\mathfrak{L}(\mathcal{H}_s)$:

$$\mathrm{w}^*-\lim_{\varepsilon \to 0} \big(\mathcal{G}_s(t, Y) - \prod_{j=1}^{s} \mathcal{G}_1(t, j)\big) g_s = 0. \tag{18}$$

Taking into account analogs of the Duhamel equations for cumulants of asymptotically perturbed groups of operators [17], in view of formula (18) we have

$$\begin{aligned} &\mathrm{w}^*-\lim_{\varepsilon \to 0} \Big(\varepsilon^{-n} \frac{1}{n!} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, j_1, \dots, j_n) \\ &- \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \sum_{i_1 \neq j_1 = 1}^{s} \mathcal{N}_{\mathrm{int}}(i_1, j_1) \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \dots \\ &\prod_{l_n \in Y \setminus (j_1, \dots, j_{n-1})} \mathcal{G}_1(t_{n-1} - t_n, l_n) \sum_{\substack{i_n \neq j_n = 1, \\ i_n, j_n \neq (j_1, \dots, j_{n-1})}}^{s} \mathcal{N}_{\mathrm{int}}(i_n, j_n) \\ &\times \prod_{l_{n+1} \in Y \setminus (j_1, \dots, j_n)} \mathcal{G}_1(t_n, l_{n+1}) \Big) g_{s-n} = 0, \end{aligned}$$

where we used notations accepted in (17) and $g_{s-n} \equiv g_{s-n}((1, \dots, s) \setminus (j_1, \dots, j_n))$, $n \geq 1$. As a result of this equality we establish the validity of statement (16) for expansion (8) of marginal observables.

If $b^0 \in \mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$, then the sequence $b(t) = (b_0, b_1(t), \dots, b_s(t), \dots)$ of limit marginal observables (17) is a generalized global solution of the Cauchy problem of the dual quantum Vlasov hierarchy

$$\frac{\partial}{\partial t} b_s(t, Y) = \sum_{j=1}^{s} \mathcal{N}(j) \, b_s(t, Y) + \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{N}_{\mathrm{int}}(j_1, j_2) \, b_{s-1}(t, Y \setminus (j_1)), \tag{19}$$

$$b_s(t)|_{t=0} = b_s^0, \quad s \geq 1, \tag{20}$$

where the infinitesimal generator $\mathcal{N}(j)$ of the group of operators $\mathcal{G}_1(t, j)$ of j particle is defined on $g_1 \in \mathfrak{L}_0(\mathcal{H})$ by formula (13). It should be noted that equations set (19) has the structure of recurrence evolution equations. We give several examples of the evolution equations of the dual quantum Vlasov hierarchy (19) in terms of operator kernels of the limit marginal observables

$$\begin{aligned} i \frac{\partial}{\partial t} b_1(t, q_1; q_1') &= -\frac{1}{2}(-\Delta_{q_1} + \Delta_{q_1'}) b_1(t, q_1; q_1'), \\ i \frac{\partial}{\partial t} b_2(t, q_1, q_2; q_1', q_2') &= -\frac{1}{2} \sum_{i=1}^{2} (-\Delta_{q_i} + \Delta_{q_i'}) b_2(t, q_1, q_2; q_1', q_2') \\ &+ \big(\Phi(q_1' - q_2') - \Phi(q_1 - q_2)\big) \big(b_1(t, q_1; q_1') + b_1(t, q_2; q_2')\big). \end{aligned}$$

We consider the mean field limit of a particular case of marginal observables, namely the additive-type marginal observables $B^{(1)}(0) = (0, B_1^{0,\varepsilon}(1), 0, \dots)$ (the k-ary marginal observable is represented by the sequence $B^{(k)}(0) = (0, \dots, 0, B_k^{0,\varepsilon}(1, \dots, k), 0, \dots)$). In case of additive-type marginal observables expansions (8) take the following form:

$$B_s^{(1)}(t, Y) = \mathfrak{A}_s(t) \sum_{j=1}^{s} B_1^{0,\varepsilon}(j), \quad s \geq 1, \tag{21}$$

where the operator $\mathfrak{A}_s(t)$ is sth-order cumulant (9) of groups of operators (10).

If for the additive-type marginal observable $B_1^{0,\varepsilon} \in \mathfrak{L}(\mathcal{H})$ it holds

$$\mathrm{w}^*-\lim_{\varepsilon \to 0} (\varepsilon^{-1} B_1^{0,\varepsilon} - b_1^0) = 0,$$

then for additive-type marginal observables (21) there exists the following mean field limit

$$\mathrm{w}^*-\lim_{\varepsilon\to 0}(\varepsilon^{-s}B_s^{(1)}(t)-b_s^{(1)}(t))=0,\quad s\geq 1,$$

where the limit additive-type marginal observable $b_s^{(1)}(t)$ is determined by a special case of expansion (17)

$$\begin{aligned} b_s^{(1)}(t,Y)=&\int_0^t dt_1\ldots\int_0^{t_{s-2}}dt_{s-1}\prod_{l_1\in Y}\mathcal{G}_1(t-t_1,l_1)\sum_{i_1\neq j_1=1}^{s}\mathcal{N}_{\mathrm{int}}(i_1,j_1)\\ &\times\prod_{l_2\in Y\setminus(j_1)}\mathcal{G}_1(t_1-t_2,l_2)\ldots\prod_{l_{s-1}\in Y\setminus(j_1,\ldots,j_{s-2})}\mathcal{G}_1(t_{s-2}-t_{s-1},l_{s-1})\\ &\times\sum_{\substack{i_{s-1}\neq j_{s-1}=1,\\ i_{s-1},j_{s-1}\neq(j_1,\ldots,j_{s-2})}}^{s}\mathcal{N}_{\mathrm{int}}(i_{s-1},j_{s-1})\prod_{l_s\in Y\setminus(j_1,\ldots,j_{s-1})}\mathcal{G}_1(t_{s-1},l_s)b_1^0(Y\setminus(j_1,\ldots,j_{s-1})). \end{aligned}\tag{22}$$

We make several examples of expansions (22) for the limit additive-type marginal observables

$$b_1^{(1)}(t,1)=\mathcal{G}_1(t,1)\,b_1^0(1),$$

$$b_2^{(1)}(t,1,2)=\int_0^t dt_1\prod_{i=1}^{2}\mathcal{G}_1(t-t_1,i)\mathcal{N}_{\mathrm{int}}(1,2)\sum_{j=1}^{2}\mathcal{G}_1(t_1,j)\,b_1^0(j).$$

Thus, for arbitrary initial states in the mean field scaling limit the kinetic evolution of quantum many-particle systems is described in terms of limit marginal observables (17) governed by the dual quantum Vlasov hierarchy (19).

Furthermore, the relation between the evolution of observables (17) and the kinetic evolution of initial states described in terms of a one-particle (marginal) density operator and correlation operators (1) is considered.

3 The quantum Vlasov-type kinetic equation with initial correlations

We assume that for the initial one-particle (marginal) density operator $F_1^{0,\varepsilon}\in\mathfrak{L}^1(\mathcal{H})$ there exists the mean field limit $\lim_{\varepsilon\to 0}\|\varepsilon F_1^{0,\varepsilon}-f_1^0\|_{\mathfrak{L}^1(\mathcal{H})}=0$, and $\lim_{\varepsilon\to 0}\|g_n^\varepsilon-g_n\|_{\mathfrak{L}^1(\mathcal{H}_n)}=0$, $n\geq 2$, then in the mean field limit the initial state is specified by the following sequence of limit operators

$$f^{(c)}=(I,f_1^0(1),g_2(1,2)\prod_{i=1}^{2}f_1^0(i),\ldots,g_n(1,\ldots,n)\prod_{i=1}^{n}f_1^0(i),\ldots).\tag{23}$$

We note that in case of initial states specified by sequence (23) the average values (mean values) of limit marginal observables (17) are determined by the limit positive continuous linear functional (15)

$$(b(t),f^{(c)})\doteq\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Tr}_{1,\ldots,n}\,b_n(t,1,\ldots,n)\,g_n(1,\ldots,n)\prod_{i=1}^{n}f_1^0(i).\tag{24}$$

For operators $b(t)\in\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $f_1^0\in\mathfrak{L}^1(\mathcal{H})$, functional (24) exists under the condition that $\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})}<\gamma$.

We shall establish the relations of mean value functional (24) represented in terms of constructed mean field asymptotics of marginal observables (17) with its representation in terms

of a solution of the quantum Vlasov-type kinetic equation with initial correlations, i.e. in case of initial states (23).

For the limit additive-type marginal observables (22) the following equality is true

$$(b^{(1)}(t),f^{(c)})=\sum_{s=0}^{\infty}\frac{1}{s!}\mathrm{Tr}_{1,\ldots,s}\,b_s^{(1)}(t,1,\ldots,s)g_s(1,\ldots,s)\prod_{i=1}^{s}f_1^0(i)=\mathrm{Tr}_1\,b_1^0(1)f_1(t,1),$$

where the operator $b_s^{(1)}(t)$ is determined by expansion (22) and the one-particle (marginal) density operator $f_1(t,1)$ is represented by the series expansion

$$\begin{aligned} f_1(t,1)=&\sum_{n=0}^{\infty}\int_0^t dt_1\ldots\int_0^{t_{n-1}}dt_n\,\mathrm{Tr}_{2,\ldots,n+1}\mathcal{G}_1^*(t-t_1,1)\mathcal{N}_{\mathrm{int}}^*(1,2)\prod_{j_1=1}^{2}\mathcal{G}_1^*(t_1-t_2,j_1)\ldots\\ &\times\prod_{i_n=1}^{n}\mathcal{G}_1^*(t_n-t_n,i_n)\sum_{k_n=1}^{n}\mathcal{N}_{\mathrm{int}}^*(k_n,n+1)\prod_{j_n=1}^{n+1}\mathcal{G}_1^*(t_n,j_n)g_{1+n}(1,\ldots,n+1)\prod_{i=1}^{n+1}f_1^0(i). \end{aligned}\tag{25}$$

In series expansion(25) the operator $\mathcal{N}_{\mathrm{int}}^*(j_1,j_2)f_n=-\mathcal{N}_{\mathrm{int}}(j_1,j_2)f_n$ is an adjoint operator to operator (12) and the group $\mathcal{G}_1^*(t,i)=\mathcal{G}_1(-t,i)$ is dual to group (10) in the sense of functional (24). For bounded interaction potentials series (25) is norm convergent on the space $\mathfrak{L}^1(\mathcal{H})$ under the condition that $t<t_0\equiv(2\,\|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)}\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$.

The operator $f_1(t)$ represented by series (25) is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations:

$$\begin{aligned} \frac{\partial}{\partial t}f_1(t,1)=&\mathcal{N}^*(1)f_1(t,1)\\ &+\mathrm{Tr}_2\,\mathcal{N}_{\mathrm{int}}^*(1,2)\prod_{i_1=1}^{2}\mathcal{G}_1^*(t,i_1)g_2(1,2)\prod_{i_2=1}^{2}(\mathcal{G}_1^*)^{-1}(t,i_2)f_1(t,1)f_1(t,2), \end{aligned}\tag{26}$$

$$f_1(t)|_{t=0}=f_1^0,\tag{27}$$

where the operator $\mathcal{N}^*(1)=-\mathcal{N}(1)$ is an adjoint operator to operator (13) in the sense of functional (24) and the group $(\mathcal{G}_1^*)^{-1}(t)=\mathcal{G}_1^*(-t)=\mathcal{G}_1(t)$ is inverse to the group $(\mathcal{G}_1^*)(t)$. This fact is proved similarly as in case of a solution of the quantum BBGKY hierarchy represented by the iteration series [13].

Thus, in case of initial states specified by one-particle (marginal) density operator (23) we establish that the dual quantum Vlasov hierarchy (19) for additive-type marginal observables describes the evolution of a quantum large particle system just as the non-Markovian quantum Vlasov-type kinetic equation with initial correlations (26).

4 The propagation of initial correlations in a mean field limit

We consider the evolution of initial correlations in a mean field scaling limit.

The property of the propagation of initial correlations is a consequence of the validity of the following equality for the mean value functional of the limit k-ary marginal observables, i.e. the sequences $b^{(k)}(0)=(0,\ldots,0,b_k^0(1,\ldots,k),0,\ldots)$ [15] at initial instant, in case of $k\geq 2$

$$\begin{aligned} (b^{(k)}(t),f^{(c)})=&\sum_{s=0}^{\infty}\frac{1}{s!}\mathrm{Tr}_{1,\ldots,s}\,b_s^{(k)}(t,1,\ldots,s)g_s(1,\ldots,s)\prod_{j=1}^{s}f_1^0(j)\\ =&\frac{1}{k!}\mathrm{Tr}_{1,\ldots,k}\,b_k^0(1,\ldots,k)\prod_{i_1=1}^{k}\mathcal{G}_1^*(t,i_1)g_k(1,\ldots,k)\prod_{i_2=1}^{k}(\mathcal{G}_1^*)^{-1}(t,i_2)\prod_{j=1}^{k}f_1(t,j), \end{aligned}\tag{28}$$

where the limit one-particle (marginal) density operator $f_1(t,j)$ is represented by series expansion (25) and therefore it is governed by the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations (26), (27).

This fact is proved similarly to the proof of a property on the propagation of initial chaos in a mean field scaling limit [18].

Therefore in case of initial states specified by sequence (1) mean field dynamics of all possible states is described in terms of the sequence $f = (I, f_1(t), f_2(t), \ldots, f_n(t), \ldots)$ of the limit marginal density operators $f_n(t,1,\ldots,n)$, $n \geq 1$, which are represented within the framework of the one-particle density operator $f_1(t)$ as follows

$$f_n(t,1,\ldots,n) = \prod_{i_1=1}^{n} \mathcal{G}_1^*(t,i_1) g_n(1,\ldots,n) \prod_{i_2=1}^{n} (\mathcal{G}_1^*)^{-1}(t,i_2) \prod_{j=1}^{n} f_1(t,j), \quad n \geq 2,$$

where the one-particle density operator $f_1(t,j)$ is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations (26),(27). In case of initial states specified by sequence (4) of the marginal correlation operators the evolution of all possible correlations is described by the following sequence of the limit marginal correlation operators

$$g_n(t,1,\ldots,n) = \prod_{i_1=1}^{n} \mathcal{G}_1^*(t,i_1) \tilde{g}_n(1,\ldots,n) \prod_{i_2=1}^{n} (\mathcal{G}_1^*)^{-1}(t,i_2) \prod_{j=1}^{n} f_1(t,j), \quad n \geq 2,$$

where the operators $\tilde{g}_n$ related to operators g_n by expansions (5).

We note that the general approach to the description of the evolution of states of quantum many-particle systems within the framework of correlation operators and marginal correlation operators was given in paper [19].

Thus, in case of the limit k-ary marginal observables solution (22) of the dual quantum Vlasov hierarchy (19) is equivalent to a property of the propagation of initial correlations for the k-particle marginal density operator in the sense of equality (28) or in other words the mean field scaling dynamics does not create correlations.

5 Conclusion and outlook

In the paper the concept of quantum kinetic equations in case of the kinetic evolution, involving correlations of particle states at initial time, for instance, correlation operators characterizing the condensed states, was considered.

This paper deals with a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics. The obtained results can be extended to quantum systems of bosons or fermions.

In case of pure states the quantum Vlasov-type kinetic equation with initial correlations (26) can be reduced to the Gross–Pitaevskii-type kinetic equation [14]. Indeed, in this case the one-particle density operator $f_1(t) = |\psi_t\rangle\langle\psi_t|$ is a one-dimensional projector onto a unit vector $|\psi_t\rangle \in \mathcal{H}$ and its kernel has the following form: $f_1(t,q,q') = \psi(t,q)\psi^*(t,q')$. Then, if we consider quantum particles, interacting by the potential which kernel $\Phi(q) = \delta(q)$ is the Dirac measure, from kinetic equation (26) we derive the Gross–Pitaevskii-type kinetic equation [20]

$$i\frac{\partial}{\partial t}\psi(t,q) = -\frac{1}{2}\Delta_q \psi(t,q) + \int dq'dq''\mathfrak{g}(t,q,q;q',q'')\psi(t,q'')\psi^*(t,q)\psi(t,q),$$

where the coupling ratio $\mathfrak{g}(t,q,q;q',q'')$ of the collision integral is the kernel of the scattering length operator $\mathcal{G}_1^*(t,1)\mathcal{G}_1^*(t,2)g_2(1,2)$. If we consider a system of quantum particles without initial correlations (7) (or (6)), then this kinetic equation is the cubic nonlinear Schrödinger equation [13].

We note also that in paper [21] it was developed one more method of the derivation of quantum kinetic equations. By means of a non-perturbative solution of the quantum BBGKY hierarchy it was established that, if initial data is completely specified by a one-particle marginal density operator (in case of initial data with correlations see paper [20]), then all possible states of quantum many-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator governed by the generalized quantum kinetic equation. The actual quantum kinetic equations can be derived from the generalized quantum kinetic equation in the appropriate scaling limit, for example, in a mean field limit [18]. We emphasize that one of the advantages of such an approach to the derivation of the quantum kinetic equations from underlying dynamics governed by the generalized quantum kinetic equation consists in an opportunity to construct the higher-order corrections to the scaling asymptotic behavior of large particle quantum systems.

References

[1] Spohn H. *Kinetic equations from Hamiltonian dynamics: Markovian limits.* Rev. Modern Phys. 1980, **53** (3), 569–615.

[2] Cercignani C., Gerasimenko V.I., Petrina D.Ya. Many-Particle Dynamics and Kinetic Equations. Kluwer Acad. Publ., The Netherlands, 1997.

[3] Gallagher I., Saint-Raymond L., Texier B. From Newton to Boltzmann: Hard Spheres and Short-Range Potentials. In: Zürich Lectures in Advanced Mathematics, 18. EMS, 2014.

[4] Pulvirenti M., Saffirio C., Simonella S. *On the validity of the Boltzmann equation for short range potentials.* Rev. Math. Phys. 2014, **26** (2), 1450001. doi:10.1142/S0129055X14500019

[5] Bogolyubov M.M. Lectures on Quantum Statistics. Problems of Statistical Mechanics of Quantum Systems. Rad. Shkola, Kyiv, 1949. (in Ukrainian)

[6] Saint-Raymond L. *Kinetic models for superfluids: a review of mathematical results.* C. R. Physique. 2004, **5**, 65–75. doi:10.1016/j.crhy.2004.01.005

[7] Lieb E., Seiringer R., Solovej J.P., Yngvason J. The mathematics of the Bose gas and its condensation. In: Oberwolfach Seminars, 34. Birkhäuser Verlag, Basel, 2005.

[8] Bao W., Cai Y. *Mathematical theory and numerical methods for Bose–Einstein condensation.* Kinet. Relat. Models 2013, **6** (1), 1–135.

[9] Semkat D., Kremp D., Bonitz M. *Kadanoff–Baym equations with initial correlations.* Phys. Rev. E. 1999, **59** (2), 1557–1562.

[10] Kadanoff L.P., Baym G. Quantum Statistical Mechanics. W.A. Benjamin, 1962.

[11] Pezzotti F., Pulvirenti M. *Mean-field limit and semiclassical expansion of quantum particle system.* Ann. Henri Poincaré 2009, **1**, 145–187. doi:10.1007/s00023-009-0404-1

[12] Fröhlich J., Graffi S., Schwarz S. *Mean-field and classical limit of many-body Schrödinger dynamics for bosons.* Comm. Math. Phys. 2007, **271**, 681–697. doi:10.1007/s00220-007-0207-5

[13] Erdős L., Schlein B., Yau H.-T. *Derivation of the cubic nonlinear Schrödinger equation from quantum dynamics of many-body systems.* Invent. Math. 2007, **167** (3), 515–614. doi:10.1007/s00222-006-0022-1

[14] Erdős L., Schlein B., Yau H.-T. *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate.* Ann. of Math. Stud. 2010, **172** (1), 291–370. doi:10.4007/annals.2010.172.291

[15] Borgioli G., Gerasimenko V.I. *Initial-value problem of the quantum dual BBGKY hierarchy.* Il Nuovo Cimento C 2010, **33 C** (1), 71–78. doi:10.1393/ncc/i2010-10564-6

[16] Gerasimenko V.I. *Heisenberg picture of quantum kinetic evolution in mean-field limit.* Kinet. Relat. Models 2011, **4** (1), 385–399. doi:10.3934/krm.2011.4.385

[17] Banasiak J., Lachowicz M. Methods of Small Parameter in Mathematical Biology. Birkhäuser, Boston, 2014.

[18] Gerasimenko V.I., Tsvir Zh.A. *Mean field asymptotics of generalized quantum kinetic equation.* Rep. Math. Phys. 2012, **70** (2), 135–147. doi:10.1016/S0034-4877(12)60034-2

[19] Gerasimenko V.I., Polishchuk D.O. *A nonperturbative solution of the nonlinear BBGKY hierarchy for marginal correlation operators.* Math. Methods Appl. Sci. 2013, **36** (17), 2311–2328. doi:10.1002/mma.2753

[20] Gerasimenko V.I., Tsvir Zh.A. *On quantum kinetic equations of many-particle systems in condensed states.* Phys. A 2012, **391** (24), 6362–6366. doi:10.1016/j.physa.2012.07.061

[21] Gerasimenko V.I., Tsvir Zh.A. *A description of the evolution of quantum states by means of the kinetic equation.* J. Phys. A: Math. Theor. 2010, **43** (48), 485203. doi:10.1088/1751-8113/43/48/485203

Received 09.03.2015

Герасименко В.І. *Новий метод виведення квантових кінетичних рівнянь з початковими кореляціями* // Карпатські матем. публ. — 2015. — Т.7, №1. — С. 38–48.

Запропоновано новий метод виведення кінетичних рівнянь з динаміки квантових систем багатьох частинок за наявності кореляцій станів частинок в початковий момент. Розвинутий підхід грунтується на описі еволюції за допомогою маргінальних спостережуваних в скейлінгових границях. В результаті побудовано власовського типу квантове кінетичне рівняння з початковими кореляціями та доведено твердження стосовно властивості поширення початкових кореляцій в ганиці самоузгодженого поля.

Ключові слова і фрази: маргінальні спостережувані, кінетичне рівняння з початковими кореляціями.

ISSN 2075-9827
Carpathian Math. Publ. 2015, **7** (1), 49–56
doi:10.15330/cmp.7.1.49-56

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.49–56

GUMENCHUK A.I.

LATERAL CONTINUITY AND ORTHOGONALLY ADDITIVE OPERATORS

We generalize the notion of a laterally convergent net from increasing nets to general ones and study the corresponding lateral continuity of maps. The main result asserts that, the lateral continuity of an orthogonally additive operator is equivalent to its continuity at zero. This theorem holds for operators that send laterally convergent nets to any type convergent nets (laterally, order or norm convergent).

Key words and phrases: orthogonally additive operator, lateral continuity.

Chernivtsi Medical College, 60 Geroiv Maidanu str., 58001, Chernivtsi, Ukraine
E-mail: anna_hostyuk@ukr.net

1 INTRODUCTION

Some versions of laterally (i.e., horizontally) continuous maps acting between vector lattices were considered in [4] and [6]. A net (x_α) in a vector lattice E in the mentioned above papers is called laterally convergent to $x \in E$ if $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ as $\alpha < \beta$ and $x_\alpha \xrightarrow{\mathrm{o}} x$. Here and in the sequel the relation $u \sqsubseteq v$ means that u is a fragment (component, in another terminology) of v, that is, $u \perp (v-u)$, and the notation $x_\alpha \xrightarrow{\mathrm{o}} x$ means that the net (x_α) order converges to x, i.e. there is a net (u_α) in E with the same index set such that $|x_\alpha - x| \le u_\alpha$ for all α, and $u_\alpha \downarrow 0$, that is, (u_α) is a decreasing (in the non-strict sense) net with zero infimum. In our opinion, the assumption $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ on the net in the above definition of the lateral convergence is too restrictive and unjustified. One of the tasks of the present note is to generalize the lateral convergence to not necessarily laterally increasing nets.

In [4] the authors considered maps that laterally convergent nets send to order convergent nets (such maps were called disjointly continuous). In [6] the maps that laterally convergent nets send to norm convergent nets in a normed space were called laterally-to-norm continuous. In both papers [4] and [6] laterally convergent nets were considered to be laterally increasing. Another task of the present paper is to analyze the relationships between different versions of lateral continuity. We provide an example of a disjointly continuous map which is not laterally continuous in the sense of new (generalized) definition of the lateral continuity. However, we do not know if there exists an orthogonally additive operator of the kind.

Due to the generalized definition of the lateral continuity, there are nontrivial nets laterally converging to zero. So, it is naturally to ask, whether the lateral continuity of a linear (or, more general, orthogonally additive) operator can be reduced to the same continuity at zero. Our mail result answer this in the affirmative.

УДК 517.982
2010 *Mathematics Subject Classification:* 47H30, 47H99.

1.1 Terminology and notation

Terminology, notation and facts on vector lattices, that are familiarly used in the paper were taken from [1]. The equality $z = x \sqcup y$ for elements x, y, z of a vector lattice E means that $z = x + y$ and $x \perp y$, that is, $|x| \wedge |y| = 0$. All vector lattices considered in the paper are assumed to be Archimedean.

For the first time orthogonally additive operators on vector lattices were considered and investigated in [4] and [5]. Let E be a vector lattice and X be a vector space. A function $T : E \to X$ is called an *orthogonally additive operator* if $T(x \sqcup y) = T(x) + T(y)$. In other words, orthogonally additive operators the sum of two disjoint elements send to the sum of their images.

An important example of a nonlinear orthogonally additive operator is the positive part x^+ of an element x in a vector lattice E. Show that, if $x \perp y$ then $(x+y)^+ = x^+ + y^+$. Using the well known properties $(u+v) \vee (u+w) = u + (v \vee w)$ [1, Theorem 2.1] and $\sup(-A) = -\inf A$ [1, p. 3] for $u, v, w \in E$ and $A \subseteq E$, taking into account that $x^+ \perp y^-$, $y^+ \perp x^-$, and that the disjoint (orthogonal) complement is a linear space [1, Theorem 3.3], we obtain $(x^+ + y^+) \wedge (x^- + y^-) = 0$, and hence

$$\begin{aligned}(x+y)^+ &= (x+y) \vee 0 = (x^+ + y^+ - x^- - y^-) \vee (x^+ + y^+ - x^+ - y^+)\\ &= x^+ + y^+ + (-(x^- + y^-) \vee -(x^+ + y^+))\\ &= x^+ + y^+ - (x^+ + y^+) \wedge (x^- + y^-) = x^+ + y^+.\end{aligned}$$

We use several times the example of a vector lattice $\mathbb{R}^\Omega$ of all functions $x : \Omega \to \mathbb{R}$ with respect to the pointwise linear operations of taking the sum and the multiplication by a scalar, and with the pointwise order: $x \le y$ if and only if $x(t) \le y(t)$ for all $t \in \Omega$. Given a subset $A \subseteq \Omega$, the symbol $\mathbf{1}_A$ denotes the characteristic function of A, that is, the function $\mathbf{1}_A : \Omega \to \mathbb{R}$ given by

$$\mathbf{1}_A(t) = \begin{cases} 1, & \text{if } t \in A,\\ 0, & \text{if } t \in \Omega \setminus A.\end{cases}$$

Definitions and necessary properties of Boolean algebras see in [2, Definition 7.9].

1.2 The lateral order

For the first time the lateral order and its properties were considered in [3]. But, as far as we know, the cited paper is not yet published. So, for convenience of the reader, propositions that we took from [3], we provide with complete proofs and citation.

Proposition 1 ([3]). *Let E be a vector lattice and $x, y \in E$.*

(1) If $x \sqsubseteq y$ then

(a) $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$,

(b) $x^+ \le y^+$ and $x^- \le y^-$,

(c) $x^- \perp y^+$ and $x^+ \perp y^-$,

(d) $|x| \sqsubseteq |y|$.

(2) $x \sqsubseteq y$ if and only if $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$.

Proof. Assume $x \sqsubseteq y$, that is, $y = x \sqcup (y - x)$. Then $y^+ = x^+ \sqcup (y-x)^+$, which implies $x^+ \le y^+$ and $(y-x)^+ = y^+ - x^+$. Hence $y^+ = x^+ \sqcup (y^+ - x^+)$, i.e., $x^+ \sqsubseteq y^+$. Analogously, $x^- \le y^-$ and $x^- \sqsubseteq y^-$. Thus, *(a)*, *(b)* and the "only if" part of item *(2)* is proved.

(c) By *(b)*, $0 \le x^- \wedge y^+ \le y^- \wedge y^+ = 0$. The second part of *(c)* is proved analogously.

(d) By *(a)*, $x^+ \perp y^+ - x^+$, and by *(c)*, $x^+ \perp y^-$. Moreover, $x^+ \perp x^-$. Hence $x^+ \perp |y| - |x|$. Analogously, $x^- \perp |y| - |x|$. The latter two relations yield $|x| \perp |y| - |x|$, that is, $|x| \sqsubseteq |y|$.

The "if" part of *(2)*. Suppose $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$. Then the first relation implies $x^+ \le y^+$. Then $0 \le x^+ \wedge y^- \le y^+ \wedge y^- = 0$, and hence $x^+ \perp y^-$. Taking into account $x^+ \perp (y^+ - x^+)$ and $x^+ \perp x^-$, one gets $x^+ \perp (y^+ - x^+ - y^- + x^-)$, i.e., $x^+ \perp (y - x)$. Analogously, $x^- \perp (y - x)$, and thus, $x \perp (y - x)$. □

Proposition 2 ([3]). *Let E be a vector lattice. Then the binary relation $\sqsubseteq$ is a partial order on E.*

Proof. For every $x \in E$ the relation $x \sqsubseteq x$ means that $x \perp 0$, which is obviously valid.

Assume $x, y \in E$ and $x \sqsubseteq y \sqsubseteq x$. Since $x \perp (y - x)$ and $y \perp (y - x)$, one has $(y - x) \perp (y - x)$, that is, $y - x = 0$. Let $x, y, z \in E$ and $x \sqsubseteq y \sqsubseteq z$. Then $x \perp (y - x)$. Moreover, by *(1) (b)* of Proposition 1 one has $|x| \le |y|$. The latter inequality together with $y \perp (z - y)$ gives $x \perp (z - y)$. Since the orthogonal complement is a linear space [1, Theorem 3.3], we obtain $x \perp (y - x) + (z - y) = z - x$, that is $x \sqsubseteq z$. □

Given any $e \in E$, by $\mathfrak{F}_e$ we denote the set of all fragments of e, $\mathfrak{F}_e = \{x \in E : x \sqsubseteq e\}$. Item *(1)* of the following proposition is very known for $e \ge 0$ [1, Theorem 3.15].

Proposition 3 ([3]). *Let E be a vector lattice and $e \in E$. Then*

(1) the set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with zero 0, unit e with respect to the operations $x \cup y = (x^+ \vee y^+) - (x^- \vee y^-)$ and $x \cap y = (x^+ \wedge y^+) - (x^- \wedge y^-)$;

(2) if $e \ge 0$ then the lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\le$, and hence the lateral supremum (infimum) of an arbitrary set $A \subseteq \mathfrak{F}_e$ equals its lattice supremum;

(3) $x \cup y$ equals the supremum, and $x \cap y$ equals the infimum of a two-point set $\{x, y\} \subseteq \mathfrak{F}_e$ with respect to the lateral order $\sqsubseteq$ both in $\mathfrak{F}_e$ and E.

Proof. *(1)* By [1, Theorem 3.15], $\mathfrak{F}_{e^+}$ and $\mathfrak{F}_{e^-}$ are Boolean algebras with zero 0, units e^+ and e^- respectively and operations $\vee$ and $\wedge$, that coincide with the lattice operations on E. Consider the direct sum $\mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$, that is, the Cartesian product $\mathfrak{F}_{e^+} \times \mathfrak{F}_{e^-}$ with zero $(0,0)$, unit (e^+, e^-) and operations $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$ and $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$. Obviously, $\mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$ is a Boolean algebra. Then the bijection $\tau : \mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-} \to \mathfrak{F}_e$ given by $\tau(x, y) = x - y$ for any $(x, y) \in \mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$ (the facts that $\tau(x, y) \in \mathfrak{F}_e$, and that τ is one-to-one follow from Proposition 1) induces the Boolean algebra structure on $\mathfrak{F}_e$. It remains to observe that τ sends $(0,0)$ to 0, (e^+, e^-) to $e^+ - e^- = e$, and the induces operations are given by the formulas given in the statement of *(1)*.

(2) Assume $e \ge 0$ and $x, y \in \mathfrak{F}_e$. By Proposition 1, $x, y \ge 0$.

Let $x \sqsubseteq y$. By *(1) (b)* of Proposition 1, we get $x \le y$.

Let $x \le y$. Then $0 \le x \wedge (e - y) \le x \wedge (e - x) = 0$, and hence $x \perp (e - y)$. Since $x \perp (e - x)$ and the disjoint complement is a linear subspace [1, Theorem 3.3], we obtain $x \perp (y - x)$, and hence $x \sqsubseteq y$.

(3) follows from *(2)* and Proposition 1. □

By Proposition 3, using the well known equality $x+y=x\vee y+x\wedge y$ [1, Theorem 1.2], we obtain the following consequence.

Corollary 1 ([3]). *Let E be a vector lattice, $e\in E$ and $x,y\sqsubseteq e$. Then $x+y=x\cup y+x\cap y$.*

Proof. The proof follows from equalities:

$$\begin{aligned} x+y &= x^+ + y^+ - (x^- + y^-) \\ &= x^+\vee y^+ + x^+\wedge y^+ - (x^-\vee y^- + x^-\wedge y^-) = x\cup y + x\cap y. \end{aligned}$$

□

In the sequel, on the Boolean algebra $\mathfrak{F}_e$ we will consider the set-theoretical operations $x\setminus y = x\cap(e-y)=x-x\cap y$ and $x\Delta y=(x\setminus y)\cup(y\setminus x)=(x\setminus y)\sqcup(y\setminus x)$.

Definition 1. *A subset A of a vector lattice E is said to be* laterally bounded *if $A\subseteq \mathfrak{F}_e$ for some $e\in E$.*

2 Lateral convergence

In this section, we generalize the lateral convergence from laterally increasing nets to arbitrary ones. All statements that are used to prove the main result are given as lemmas, however they could be of their own interest. By a laterally converging net in a vector lattice we mean any laterally bounded order converging net. But not only such nets. The point is that, by attaching of several new elements to a laterally bounded net, one can spoil the lateral boundedness, however, by the idea of convergence, this should not affect the lateral convergence. Taking this into account, we give the next definition.

Definition 2. *An order converging net (x_α) to an element x of a vector lattice E, so that there is an index α_0 such that the net $(x_\alpha)_{\alpha\geq\alpha_0}$ is laterally bounded, is said to be* laterally converging *to x, and the element x is called the* lateral limit *of (x_α). The notation $x_\alpha\xrightarrow{\text{lat}}x$ means that the net (x_α) laterally converges to x. In the particular case, where $x_\alpha\sqsubseteq x_\beta$ for any $\alpha<\beta$, the laterally convergent net (x_α) is called* up-laterally convergent *to its lateral limit*[1].

It is interesting to observe that the lateral limit is laterally bounded by the same element as the net itself. This follows from the next statement.

Lemma 1. *Let E be a vector lattice and $e\in E$. Then the set $\mathfrak{F}_e$ is order closed.*

Proof. Let $x_\alpha\xrightarrow{\text{o}}x$, where $x_\alpha\in\mathfrak{F}_e$ and $x\in E$. Show that $x\sqsubseteq e$. By the order continuity of the lattice operations, $0=|x_\alpha|\wedge|e-x_\alpha|\xrightarrow{\text{o}}|x|\wedge|e-x|$, and hence, $|x|\wedge|e-x|=0$. □

As an immediate consequence of Lemma 1 we obtain the following fact.

Lemma 2. *Let E be a vector lattice, $e\in E$ and $x_\alpha\xrightarrow{\text{lat}}x$, where $x\in E$ and $x_\alpha\sqsubseteq e$ for all $\alpha\geq\alpha_0$. Then $x\sqsubseteq e$.*

[1] Recall that exactly these nets in [4] and [6] were said to be laterally convergent.

We say that a subset A of a vector lattice E is *laterally closed* if the lateral limit of any net from A belongs to A. Using this terminology, Lemma 2 asserts that, for any $e\in E$ the set $\mathfrak{F}_e$ is laterally closed.

Next we show that, in the definition of the lateral convergence, one can choose a majorizing net to be laterally bounded.

Proposition 4. *Let E be a Dedekind complete vector lattice, $e\in E$, $x_\alpha\xrightarrow{\text{lat}}x$, where $x\in E$ and $x_\alpha\sqsubseteq e$ for all $\alpha\geq\alpha_0$. Then there is a net (v_α) with the same index set such that $v_\alpha\sqsubseteq|e|$ and $|x_\alpha-x|\sqsubseteq v_\alpha$ for all $\alpha\geq\alpha_0$ and $v_\alpha\downarrow 0$.*

For the proof, we need the following lemma.

Lemma 3. *Let E be a vector lattice, $e\in E$ and $x,y\sqsubseteq e$. Then $|x-y|=|x\Delta y|\sqsubseteq|e|$.*

Proof of Lemma 3. Subtracting from the equality $x=(x\setminus y)\sqcup(x\cap y)$ the equality $y=(y\setminus x)\sqcup(x\cap y)$, we obtain $x-y=(x\setminus y)-(y\setminus x)$. Since $(x\setminus y)\perp(y\setminus x)$, by the orthogonal additivity of the positive part of an element and Corollary 1, we obtain

$$|x-y|=|x\setminus y|+|y\setminus x|=|(x\setminus y)+(y\setminus x)|=|(x\setminus y)\cup(y\setminus x)|=|x\Delta y|.$$

Since $x\Delta y\sqsubseteq e$, by item (1)(d) of Proposition 1 we get $|x\Delta y|\sqsubseteq|e|$. □

Proof of Proposition 4. Let (u_α) be a net in E such that $|x_\alpha-x|\leq u_\alpha\downarrow 0$. For every α we set $v_\alpha=\bigvee_{\beta\geq\alpha}|x_\beta-x|$. The supremum exists because $|x_\beta-x|\leq 2e$ for all β and E is Dedeking complete. By Lemma 3, $|x_\beta-x|\sqsubseteq|e|$ for all β. By (2) of Proposition 3, v_α equals the lateral supremum of the net $(|x_\beta-x|)_{\beta\geq\alpha}$. Hence $v_\alpha\sqsubseteq|e|$. The inequality $|x_\alpha-x|\leq v_\alpha$ for all α follows from the construction of v_α. Finally, the condition $v_\alpha\downarrow 0$ follows from

$$0\leq v_\alpha\leq\bigvee_{\beta\geq\alpha}u_\beta=u_\alpha\downarrow 0.$$

□

Lemma 4. *Let E be a vector lattice, (x_α) a net in E and $x\in E$. Then the following assertions are equivalent:*

(i) $x_\alpha\xrightarrow{\text{lat}}x$;

(ii) $x_\alpha^+\xrightarrow{\text{lat}}x^+$, $x_\alpha^-\xrightarrow{\text{lat}}x^-$ *and $(x_\alpha)_{\alpha\geq\alpha_0}$ is laterally bounded for some α_0;*

(iii) The set $\{x\}\cup\{x_\alpha:\alpha\geq\alpha_0\}$ is laterally bounded and $x_\alpha\Delta x\xrightarrow{\text{lat}}0$.

Moreover, each of (i)–(iii) implies $|x_\alpha|\xrightarrow{\text{lat}}|x|$.

Proof. (i) ⇔ (ii) The equivalence of $x_\alpha\xrightarrow{\text{o}}x$ and the conditions $x_\alpha^+\xrightarrow{\text{o}}x^+$, $x_\alpha^-\xrightarrow{\text{o}}x^-$ is easily seen. It remains to observe that, the lateral boundedness of (x_α) implies that of the nets (x_α) and (x_α) by Proposition 1.

(i) ⇒ (iii) Assume $x_\alpha\xrightarrow{\text{lat}}x$. By Lemma 2, there is $e\in E$ such that $x,x_\alpha\sqsubseteq e$ for all $\alpha\geq\alpha_0$. Then the net $(x_\alpha\Delta x)_{\alpha\geq\alpha_0}$ is laterally bounded by e. Moreover, by Lemma 3, $|x_\alpha\Delta x|=|x_\alpha-x|$, and hence $x_\alpha\Delta x\xrightarrow{\text{o}}0$.

(iii) ⇒ (i) directly follows from Lemma 3.

It remains to observe that the condition $|x_\alpha|\xrightarrow{\text{lat}}|x|$ follows from *(1) (d)* of Proposition 1. □

Remark that the assumption of lateral boundedness of the set $\{x\} \cup \{x_\alpha : \alpha \geq \alpha_0\}$ in *(iii)* serves for the elements $x_\alpha \Delta x$ to be well defined, and the implication *(ii)* $\Rightarrow$ *(i)* may fail to be valid if one removes the assumption of lateral boundedness of the net (x_α) in *(ii)*, as the following example shows.

Example 1. *There exist a vector lattice E, a sequence (x_n) in E and an element $x \in E$ such that $x_n^+ \xrightarrow{\text{lat}} x^+$, $x_n^- \xrightarrow{\text{lat}} x^-$, but for every $n_0 \in \mathbb{N}$ the sequence $(x_n)_{n \geq n_0}$ is not laterally bounded, and hence, (x_n) laterally diverges.*

Proof. Indeed, consider the vector lattice $E = \mathbb{R}^{\mathbb{R}}$ with the pointwise order and the sequence (x_n) in E, given by

$$x_n(t) = \begin{cases} 1, & \text{if } t \in (-\infty, \frac{1}{n}], \\ -1, & \text{if } t \in (\frac{1}{n}, +\infty). \end{cases}$$

It is a simple technical exercise to show that the sequence (x_n) order converges to

$$x(t) = \begin{cases} 1, & \text{if } t \in (-\infty, 0], \\ -1, & \text{if } t \in (0, +\infty), \end{cases}$$

however, the sequence $(x_n)_{n \geq n_0}$ is not laterally bounded for all $n_0 \in \mathbb{N}$. On the other hand, $x_n^+ = \mathbf{1}_{(-\infty, \frac{1}{n}]} \xrightarrow{\text{o}} \mathbf{1}_{(-\infty, 0]}$. Since $x_n^+ \sqsubseteq \mathbf{1}_{(-\infty, 1]}$ for al $n \in \mathbb{N}$, one has that $x_n^+ = \mathbf{1}_{(-\infty, \frac{1}{n}]} \xrightarrow{\text{lat}} \mathbf{1}_{(-\infty, 0]}$. Analogously, $x_n^- = \mathbf{1}_{(\frac{1}{n}, +\infty)} \xrightarrow{\text{lat}} \mathbf{1}_{(0, +\infty)}$. □

3 Lateral continuity

In this section we study versions of continuity connected to the lateral convergence.

Definition 3. *Let E, F be vector lattices. A function $f : E \to F$ is said to be:*

(L-L) laterally continuous *at a point $x \in E$ if for any net (x_α) in E the relation $x_\alpha \xrightarrow{\text{lat}} x$ implies $f(x_\alpha) \xrightarrow{\text{lat}} f(x)$;*

(L-O) laterally-to-order continuous *at a point $x \in E$ if for any net (x_α) in E the relation $x_\alpha \xrightarrow{\text{lat}} x$ implies $f(x_\alpha) \xrightarrow{\text{o}} f(x)$.*

Definition 4. *Let E be a vector lattice and F a normed space. A function $f : E \to F$ is said to be*

(L-N) laterally-to-norm continuous *at a point $x \in E$ if for any net (x_α) in E the condition $x_\alpha \xrightarrow{\text{lat}} x$ yields $\|f(x_\alpha) - f(x)\| \to 0$.*

Following the terminology of [4], a map $f : E \to F$ acting from a vector lattice E to a vector lattice or a normed space F is said to be *disjointly laterally* (*disjointly order* or *disjointly norm*) *continuous* at a point $x \in E$ if for every net (x_α) in E up- laterally converging to x the net $(f(x_\alpha))$ laterally (order or norm, respectively) converges to $f(x)$ in F. The corresponding type of convergence we denote by *(DL-L)*, *(DL-O)* or *(DL-N)*.

We say that a function $f : E \to F$ is *continuous* in some of the senses (*(L-L)*, *(L-O)*, *(L-N)*, *(DL-L)*, *(DL-O)* or *(DL-N)*), if f is continuous in the same sense at any point $x \in E$.

Notice that the generalization of the notion of a laterally convergent net from up-laterally convergent nets to arbitrary nets may affect the lateral continuity at a fixed point. Indeed, if a net (x_α) in a vector lattice E up-laterally converges to zero then $x_\alpha = 0$ for all α. Hence, an arbitrary map $f : E \to F$ up-laterally convergent to zero nets sends to convergent nets in any sense. So, it is not a big deal to provide an example where the same happen at a nonzero point $x_0 \in E$ (say, at a point x_0 which is an atom in E, that is, the only fragments of x_0 are 0 and x_0 itself). It is clear that not every map acting from $E = \mathbb{R}^{\mathbb{R}}$ to a nontrivial vector lattice or a normed space is continuous in any of the senses *(L-L)*, *(L-O)* or *(L-N)* at x_0. For instance, the one given by $f(x_0) = 0$ and $f(x) = y_0 \neq 0$ for all $x \in E \setminus \{x_0\}$. Indeed, the sequence $x_n = \mathbf{1}_{[0, \frac{1}{n}]}$ laterally converges to x_0, however $f(x_n) = y_0 \not\to 0$ in any of the senses *(L-L)*, *(L-O)* or *(L-N)*.

The following theorem, which is the main result, in particular, asserts that the lateral continuity of an orthogonally additive operator is equivalent to its lateral continuity just at zero.

Theorem 1. *Let E be a vector lattice, F a vector lattice or a normed space, $T : E \to F$ an orthogonally additive operator. Let X be one of the letters L, O or N. Then the following assertions are equivalent:*

(1) T is (L-X) *continuous;*

(2) T is (L-X) *continuous at zero.*

Proof. The implication *(1)* $\Rightarrow$ *(2)* is obvious. Prove *(2)* $\Rightarrow$ *(1)*. Let (x_α) be a net in E, $x \in E$ and $x_\alpha \xrightarrow{\text{lat}} x$. Choose $e \in E$ and an index α_0 so that $x_\alpha \sqsubseteq e$ as $\alpha \geq \alpha_0$. Then, by Lemma 2, $x \sqsubseteq e$. Next, Lemma 4 implies that $x_\alpha \Delta x \xrightarrow{\text{lat}} 0$. Let (u_α) be a net in E such that $|x_\alpha \Delta x| \leq u_\alpha \downarrow 0$. Taking into account that $x_\alpha \Delta x = (x_\alpha \setminus x) \cup (x \setminus x_\alpha)$, we obtain $|x_\alpha \setminus x| \leq u_\alpha \downarrow 0$ and $|x \setminus x_\alpha| \leq u_\alpha \downarrow 0$. Then $x_\alpha \setminus x \xrightarrow{\text{o}} 0$ and $x \setminus x_\alpha \xrightarrow{\text{o}} 0$, and hence, $x_\alpha \setminus x \xrightarrow{\text{lat}} 0$ and $x \setminus x_\alpha \xrightarrow{\text{lat}} 0$. By the *(L-X)*-continuity at zero, $T(x_\alpha \setminus x) \to 0$ and $T(x \setminus x_\alpha) \to 0$ in the sense of X-convergence, because $T(0) = 0$ (as T is orthogonally additive). Since $x_\alpha = (x_\alpha \setminus x) \sqcup (x_\alpha \cap x)$, by the orthogonal additivity of T,

$$T(x_\alpha) = T(x_\alpha \setminus x) + T(x_\alpha \cap x). \tag{1}$$

Analogously,

$$T(x) = T(x \setminus x_\alpha) + T(x \cap x_\alpha). \tag{2}$$

Subtracting from (1) the equality (2), we obtain $T(x_\alpha) - T(x) = T(x_\alpha \setminus x) - T(x \setminus x_\alpha) \to 0$ in the sense of X. □

The following example shows that, the notion of lateral continuity changes when replacing the up-laterally convergent nets with arbitrary lateral converging nets.

Recall that, following [4], a map $f : E \to F$ between vector lattices E and F is called disjointly continuous if for every $x \in E$ and every up-laterally convergent net (x_α) the condition $x_\alpha \xrightarrow{\text{lat}} x$ implies $f(x_\alpha) \xrightarrow{\text{o}} f(x)$.

Example 2. *There exist vector lattices E, F and a disjointly continuous map $f : E \to F$ which is not laterally-to-order continuous.*

Proof. Set $E = \mathbb{R}^{[0,1]}$, $F = \mathbb{R}^{[0,2]}$ and define a map $f : E \to F$ by $f(0) = 0$ and $f(x) = x + \mathbf{1}_{(1,2]}$ for $x \in E \setminus \{0\}$. Then f is disjointly continuous at zero, because all up-laterally convergent to zero nets consist of zero elements, and the disjoint continuity of f at any nonzero point is obvious. Show that f is not laterally continuous at zero. Indeed, for the sequence $x_n = \mathbf{1}_{(0,\frac{1}{n})}$, $n = 1, 2, \dots$ one has $x_n \xrightarrow{\text{lat}} 0$, and nevertheless, $f(x_n) = x_n + \mathbf{1}_{(1,2]} \xrightarrow{\text{lat}} \mathbf{1}_{(1,2]} \neq 0 = f(0)$. □

We do not know if there is an orthogonally additive operator with the same properties.

Problem. *Do there exist vector lattices E, F and an orthogonally additive operator $T : E \to F$ which is not laterally-to-order continuous?*

Remark that any other version of Theorem 1 holds true in which instead of the convergence in the sense X one considers another convergence (say, topological), which has the property of uniqueness of limit and such that the sum of two convergent nets converges to the sum of their limits.

ACKNOWLEDGMENTS

The author thanks her scientific advisor Popov M. M. for setting problems, helpful remarks and revision of English.

REFERENCES

[1] Aliprantis C.D., Burkinshaw O. Positive operators. Springer, Dordrecht, 2006.

[2] Jech Th. Set Theory. Springer, Berlin, 2003.

[3] Mykhaylyuk V., Pliev M., Popov M. *Laterally ordered sets and orthogonally additive operators.* Preprint.

[4] Mazón J.M., Segura S.L. *Order bounded ortogonally additive operators.* Rev. Roumaine Math. Pures Appl. 1990, **35** (4), 329–353.

[5] Mazón J.M., Segura S.L. *Uryson operators.* Rev. Roumaine Math. Pures Appl. 1990, **35** (5), 431–449.

[6] Pliev M.A., Popov M.M. *Narrow orthogonally additive operators.* Positivity 2014, **18** (4), 641–667. doi:10.1007/s11117-013-0268-y

Received 22.01.2015

Гуменчук А.І. *Латеральна неперервність та ортогонально адитивні оператори* // Карпатські математ. публ. — 2015. — Т.7, №1. — C. 49–56.

Ми узагальнюємо поняття латерально збіжної сітки зі зростаючих сіток на довільні та вивчаємо відповідну латеральну неперервність відображень. Основний результат стверджує, що латеральна неперервність ортогонально адитивного оператора еквівалентна до його латеральної неперервності в нулі. Ця теорема має місце для операторів, що переводять латерально збіжні сітки у сітки, які збігаються в будь-якому розумінні (латерально, порядково чи за нормою).

Ключові слова і фрази: ортогонально адитивний оператор, латеральна неперервність.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 57–65
doi:10.15330/cmp.7.1.57-65

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.57–65

GUPTA P.

INDEX OF PSEUDO-PROJECTIVELY SYMMETRIC SEMI-RIEMANNIAN MANIFOLDS

The index of $\tilde{\nabla}$-pseudo-projectively symmetric and in particular for $\tilde{\nabla}$-projectively symmetric semi-Riemannian manifolds, where $\tilde{\nabla}$ is Ricci symmetric metric connection are discussed.

Key words and phrases: metric connection, pseudo-projective curvature tensor, projective curvature tensor, semi-Riemannian manifold, index of a manifold.

Department of Mathematics and Statistics, Dr. Harisingh Gour University, Sagar, Madhya Pradesh, 470003, India
E-mail: punam2101@gmail.com

INTRODUCTION

In 1923, Eisenhart [2] obtained the condition for the existence of a second order parallel symmetric tensor in a Riemannian manifold and proved that if a Riemannian manifold admits a second order parallel symmetric tensor other than a constant multiple of the Riemannian metric, then it is reducible. In 1925, Levy [9] gave the necessary and sufficient condition for the existence of second order parallel symmetric tensors and proved that a second order parallel symmetric non-singular tensor in a real space form is always proportional to the Riemannian metric. After that Sharma [13] improved the result of Levy and proved that any second order parallel tensor (not necessarily symmetric) in a real space form of dimension greater than 2 is proportional to the Riemannian metric. Later in 1939, Thomas [17] defined and studied the index of a Riemannian manifold. A set of metric tensors (i.e. symmetric non-degenerate parallel $(0,2)$ tensor field on the differentiable manifold) $\{H_1, \dots, H_\ell\}$ is said to be *linearly independent* if

$$c_1 H_1 + \dots + c_\ell H_\ell = 0, \qquad c_1, \dots, c_\ell \in \mathbf{R},$$

implies that $c_1 = \dots = c_\ell = 0$.

The set of metric tensors $\{H_1, \dots, H_\ell\}$ is said to be a complete set if any metric tensor H can be written as

$$H = c_1 H_1 + \dots + c_\ell H_\ell, \qquad c_1, \dots, c_\ell \in \mathbf{R}.$$

More precisely, the number of linearly independent metric tensors in a complete set of metric tensors of a Riemannian manifold is called the index of the Riemannian manifold [17, p. 413]. Therefore the existence of a second order parallel symmetric tensor is very closely related with the index of Riemannian manifolds. Then in 1968, Levine and Katzin [8] proved that the index of an n-dimensional conformally flat manifold is $n(n+1)/2$ or 1 according as it is a

УДК 514.7
2010 *Mathematics Subject Classification:* 53B05.
This work is supported by UGC-BSR Start-up Grant No. F. 30-12/2014(BSR).

flat manifold or a manifold of non-zero constant curvature. In 1981, Stavre [14] proved that if the index of an n-dimensional conformally symmetric Riemannian manifold (except the four cases of being conformally flat, of constant curvature, an Einstein manifold or with covariant constant Einstein tensor) is greater than 1, then it must be between 2 and $n+1$. In 1982, Starve and Smaranda [16] found the index of a conformally symmetric Riemannian manifolds with respect to a semi-symmetric metric connection of Yano [22]. In the recent paper [18] author and Tripathi studied the index of quasi-conformally symmetric, conformally symmetric and concircularly symmetric semi-Riemannian manifolds with respect to any metric connection and discussed some applications.

The index of the conformally flat and conformally symmetric (with respect to the Levi-Civita connection, semi-symmetric metric connection of Yano [22] and metric connection) (semi-)Riemannian manifolds were studied by many authors [8, 14, 16, 18]. Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view and the pseudo-projective curvature tensor is a generalized case of projective curvature tensor. A real space form is always pseudo-projectively flat and a pseudo-projectively flat manifold is always pseudo-projectively symmetric. But the converse is not true in both cases. The study of manifolds with semi-Riemannian metrics is of interest from the stand point of physics and relativity and have been studied by several authors. Motivated by these studies, in this paper we study the index of pseudo-projectively symmetric and projectively symmetric semi-Riemannian manifolds with respect to the metric connection $\tilde{\nabla}$. The paper is organized as follows: In Section 1, we give the preliminaries about the index of a semi-Riemannian manifold and Ricci-symmetric metric connection. In Section 2, the definition of the pseudo-projective curvature tensor in terms of projective curvature tensor and concircular curvature tensor with respect to a metric connection $\tilde{\nabla}$ are given. We also obtain a complete classification of $\tilde{\nabla}$-pseudo-projective flat (in particular, pseudo-projective flat) manifolds. In Section 3, we find out the index of $\tilde{\nabla}$-pseudo-projectively symmetric and $\tilde{\nabla}$-projectively symmetric semi-Riemannian manifolds. In the last section, some applications in theory of relativity are discussed.

1 PRELIMINARIES

Let M be an n-dimensional differentiable manifold. Let $\tilde{\nabla}$ be a linear connection in M. Then torsion tensor $\tilde{T}$ and curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X, \quad \tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z.$$

By a semi-Riemannian metric [10] on M, we understand a non-degenerate symmetric $(0,2)$ tensor field g. In [17], a semi-Riemannian metric is called a metric tensor, a positive definite symmetric $(0,2)$ tensor field, that is, Riemannian metric is called a fundamental metric tensor and a symmetric $(0,2)$ tensor field g of rank less than n is called a degenerate metric tensor.

Let (M,g) be an n-dimensional semi-Riemannian manifold. A linear connection $\tilde{\nabla}$ in M is called a metric connection with respect to the semi-Riemannian metric g if $\tilde{\nabla}g = 0$. If the torsion tensor of the metric connection $\tilde{\nabla}$ is zero, then it becomes Levi-Civita connection ∇, which is unique by the fundamental theorem of Riemannian geometry. If the torsion tensor of the metric connection $\tilde{\nabla}$ is not zero, then it is called a Hayden connection [6, 23]. Semi-symmetric metric connections [22] and quarter symmetric metric connections [4] are some well known examples of Hayden connections.

For a metric connection $\tilde{\nabla}$ in an n-dimensional semi-Riemannian manifold (M,g), the curvature tensor $\tilde{R}$ with respect to the $\tilde{\nabla}$ satisfies the following conditions

$$\tilde{R}(X,Y,Z,V) + \tilde{R}(Y,X,Z,V) = 0, \tag{1}$$

$$\tilde{R}(X,Y,Z,V) + \tilde{R}(X,Y,V,Z) = 0, \tag{2}$$

where

$$\tilde{R}(X,Y,Z,V) = g(\tilde{R}(X,Y)Z,V).$$

Let $\{e_1,\ldots,e_n\}$ be any orthonormal basis of vector fields in the manifold M. The Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ of the semi-Riemannian manifold with respect to the metric connection $\tilde{\nabla}$ is defined by

$$\tilde{S}(X,Y) = \sum_{i=1}^{n}\tilde{R}(e_i,X,Y,e_i), \tilde{r} = \sum_{i=1}^{n}\tilde{S}(e_i,e_i).$$

The Ricci operator $\tilde{Q}$ with respect to the metric connection $\tilde{\nabla}$ is defined by

$$\tilde{S}(X,Y) = g(\tilde{Q}X,Y).$$

Define

$$\tilde{e}X = \tilde{Q}X - \frac{\tilde{r}}{n}X$$

and

$$\tilde{E}(X,Y) = g(\tilde{e}X,Y).$$

Then

$$\tilde{E} = \tilde{S} - \frac{\tilde{r}}{n}g.$$

The $(0,2)$ tensor $\tilde{E}$ is known as tensor of Einstein [15] with respect to the metric connection $\tilde{\nabla}$. $\tilde{S}$ is symmetric if and only if $\tilde{E}$ is symmetric.

Definition 1 ([18]). *A metric connection $\tilde{\nabla}$ with symmetric Ricci tensor $\tilde{S}$ is called a Ricci-symmetric metric connection.*

For more details about Ricci-symmetric metric connection see [18].

Definition 2 ([18]). *Let (M,g) be an n-dimensional semi-Riemannian manifold equipped with a metric connection $\tilde{\nabla}$. A symmetric $(0,2)$ tensor field H, which is covariantly constant with respect to $\tilde{\nabla}$, is called a special quadratic first integral (for brevity SQFI) [7] with respect to $\tilde{\nabla}$. The semi-Riemannian metric g is always an SQFI. A set of SQFI tensors $\{H_1,\ldots,H_\ell\}$ with respect to $\tilde{\nabla}$ is said to be linearly independent if*

$$c_1H_1 + \cdots + c_\ell H_\ell = 0, \qquad c_1,\ldots,c_\ell \in \mathbf{R},$$

implies that $c_1 = \cdots = c_\ell = 0$.

The set $\{H_1,\ldots,H_\ell\}$ is said to be a complete set if any SQFI tensor H with respect to $\tilde{\nabla}$ can be written as $H = c_1H_1 + \cdots + c_\ell H_\ell$, $\quad c_1,\ldots,c_\ell \in \mathbf{R}$.

The index [17] of the manifold M with respect to $\tilde{\nabla}$, denoted by $i_{\tilde{\nabla}}$, is defined as the number ℓ of members in a complete set $\{H_1,\ldots,H_\ell\}$. Hence the index $i_{\tilde{\nabla}}$ of the manifold M with respect to the metric connection $\tilde{\nabla}$ is the maximum number of linearly independent SQFI in a complete set of SQFI.

2 Pseudo-projective curvature tensor

Let (M, g) be an n-dimensional $(n > 2)$ semi-Riemannian manifold equipped with a metric connection $\widetilde{\nabla}$. The projective curvature tensor $\widetilde{\mathcal{P}}$ with respect to the $\widetilde{\nabla}$ is defined by [3, p. 90]

$$\widetilde{\mathcal{P}}(X,Y,Z,V) = \widetilde{R}(X,Y,Z,V) - \frac{1}{n-1}(\widetilde{S}(Y,Z)g(X,V) - \widetilde{S}(X,Z)g(Y,V)), \tag{3}$$

and the concircular curvature tensor $\widetilde{\mathcal{Z}}$ with respect to $\widetilde{\nabla}$ is defined by ([21], [24, p. 87])

$$\widetilde{\mathcal{Z}}(X,Y,Z,V) = \widetilde{R}(X,Y,Z,V) - \frac{\widetilde{r}}{n(n-1)}(g(Y,Z)g(X,V) - g(X,Z)g(Y,V)). \tag{4}$$

As a generalization of the notion of projective curvature tensor and concircular curvature tensor, the pseudo-projective curvature tensor $\widetilde{\mathcal{P}}_*$ with respect to $\widetilde{\nabla}$ is defined by [12]

$$\begin{aligned}\widetilde{\mathcal{P}}_*(X,Y,Z,V) = {} & a\widetilde{R}(X,Y,Z,V) \\ & + b\left(\widetilde{S}(Y,Z)g(X,V) - \widetilde{S}(X,Z)g(Y,V)\right) \\ & - \frac{\widetilde{r}}{n}\left(\frac{a}{n-1} + b\right)(g(Y,Z)g(X,V) - g(X,Z)g(Y,V)),\end{aligned} \tag{5}$$

where a and b are constants. In fact, we have

$$\widetilde{\mathcal{P}}_*(X,Y,Z,V) = -(n-1)b\widetilde{\mathcal{P}}(X,Y,Z,V) + (a + (n-1)b)\widetilde{\mathcal{Z}}(X,Y,Z,V).$$

Since, there is no restrictions for manifolds if $a = 0$ and $b = 0$, therefore it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. From (5) it is clear that if $a = 1$ and $b = -1/(n-1)$, then $\widetilde{\mathcal{P}}_* = \widetilde{\mathcal{P}}$; and if $a = 1$ and $b = 0$, then $\widetilde{\mathcal{P}}_* = \widetilde{\mathcal{Z}}$.

Now, we need the following

Definition 3. *A semi-Riemannian manifold (M, g) equipped with a metric connection $\widetilde{\nabla}$ is said to be:*

(a) *$\widetilde{\nabla}$-pseudo-projectively flat if $\widetilde{\mathcal{P}}_* = 0$;*

(b) *$\widetilde{\nabla}$-projectively flat if $\widetilde{\mathcal{P}} = 0$;*

(c) *$\widetilde{\nabla}$-concircularly flat if $\widetilde{\mathcal{Z}} = 0$.*

In particular, with respect to the Levi-Civita connection ∇, $\widetilde{\nabla}$-pseudo-projectively flat, $\widetilde{\nabla}$-projectively flat and $\widetilde{\nabla}$-concircularly flat become simply pseudo-projectively flat, projectively flat and concircularly flat respectively.

Definition 4. *A semi-Riemannian manifold (M, g) equipped with a metric connection $\widetilde{\nabla}$ is said to be:*

(a) *$\widetilde{\nabla}$-pseudo-projectively symmetric if $\widetilde{\nabla}\widetilde{\mathcal{P}}_* = 0$;*

(b) *$\widetilde{\nabla}$-projectively symmetric if $\widetilde{\nabla}\widetilde{\mathcal{P}} = 0$;*

(c) *$\widetilde{\nabla}$-concircularly symmetric if $\widetilde{\nabla}\widetilde{\mathcal{Z}} = 0$.*

In particular, with respect to the Levi-Civita connection ∇, $\widetilde{\nabla}$-pseudo-projectively symmetric, $\widetilde{\nabla}$-projectively symmetric and $\widetilde{\nabla}$-concircularly symmetric become simply pseudo-projectively symmetric, projectively symmetric and concircularly symmetric respectively.

Theorem 1. *Let M be a semi-Riemannian manifold of dimension n greater than 2. Then M is $\widetilde{\nabla}$-pseudo-projectively flat if and only if one of the following statement is true:*

(i) *$a + (n-1)b = 0$, $a \neq 0 \neq b$ and M is $\widetilde{\nabla}$-projectively flat;*

(ii) *$a + (n-1)b \neq 0$, $a \neq 0$, M is $\widetilde{\nabla}$-projectively flat and $\widetilde{\nabla}$-concircularly flat;*

(iii) *$a + (n-1)b \neq 0$, $a = 0$ and Ricci tensor $\widetilde{S}$ with respect to $\widetilde{\nabla}$ satisfies*

$$\widetilde{S} - \frac{\widetilde{r}}{n}g = 0, \tag{6}$$

where $\widetilde{r}$ is the scalar curvature with respect to $\widetilde{\nabla}$.

Proof. Using $\widetilde{\mathcal{P}}_* = 0$ in (5) we get

$$\begin{aligned}0 = {} & a\widetilde{R}(X,Y,Z,V) + b(\widetilde{S}(Y,Z)g(X,V) - \widetilde{S}(X,Z)g(Y,V)) \\ & - \frac{\widetilde{r}}{n}\left(\frac{a}{n-1} + b\right)(g(Y,Z)g(X,V) - g(X,Z)g(Y,V)),\end{aligned} \tag{7}$$

from which we obtain

$$(a + (n-1)b)\left(\widetilde{S} - \frac{\widetilde{r}}{n}g\right) = 0. \tag{8}$$

Case 1. $a + (n-1)b = 0$ and $a \neq 0 \neq b$. Then from (5) and (3), it follows that $(n-1)b\widetilde{\mathcal{P}} = 0$, which gives $\widetilde{\mathcal{P}} = 0$. This gives the statement (i).

Case 2. $a + (n-1)b \neq 0$ and $a \neq 0$. Then from (8), we have

$$\widetilde{S}(Y,Z) = \frac{\widetilde{r}}{n}g(Y,Z). \tag{9}$$

Using (9) in (7), we get

$$a(\widetilde{R}(X,Y,Z,V) - \frac{\widetilde{r}}{n(n-1)}(g(Y,Z)g(X,V) - g(X,Z)g(Y,V))) = 0. \tag{10}$$

Since $a \neq 0$, then by (4), we get $\widetilde{\mathcal{Z}} = 0$ and by using (10), (9) in (3), we get $\widetilde{\mathcal{P}} = 0$. This gives the statement (ii).

Case 3. $a + (n-2)b \neq 0$ and $a = 0$, we get (6). This gives the statement (iii). Converse is true in all cases. □

Corollary 1. [19] *Let M be a semi-Riemannian manifold of dimension n greater than 2. Then M is pseudo-projectively flat if and only if one of the following statement is true:*

(i) *$a + (n-1)b = 0$, $a \neq 0 \neq b$ and M is projectively flat;*

(ii) *$a + (n-1)b \neq 0$, $a \neq 0$, M is real space form;*

(iii) *$a + (n-1)b \neq 0$, $a = 0$ and M is Einstein manifold.*

3 INDEX OF PSEUDO-PROJECTIVE SYMMETRIC MANIFOLDS

Let (M,g) be an n-dimensional semi-Riemannian manifold equipped with the metric connection $\widetilde{\nabla}$ and $\widetilde{R}$ be the curvature tensor of M with respect to the metric connection $\widetilde{\nabla}$. The integrability condition for the SQFI H is given by

$$H((\widetilde{\nabla}_U\widetilde{R})(X,Y)Z,V)+H(Z,(\widetilde{\nabla}_U\widetilde{R})(X,Y)V)=0. \tag{11}$$

Therefore, the solutions H of (11) is closely related to the index of pseudo-projectively symmetric and projectively symmetric semi-Riemannian manifolds with respect to the $\widetilde{\nabla}$.

Lemma 1. *If (M,g) be an n-dimensional semi-Riemannian $\widetilde{\nabla}$-pseudo-projectively symmetric manifold and $n>2, b\neq 0$. Then*

$$\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})=0,$$

where U is an arbitrary vector field.

Proof. Using (1) in (5), we get

$$\begin{aligned}\widetilde{\mathcal{P}}_*(X,Y,Z,V)&=a\widetilde{R}(X,Y,Z,V)+b(\widetilde{E}(Y,Z)g(X,V)-\widetilde{E}(X,Z)g(Y,V))\\&\quad-\frac{a\widetilde{r}}{n(n-1)}(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)).\end{aligned} \tag{12}$$

Taking the covariant derivative of (12) and using $\widetilde{\nabla}_U\widetilde{\mathcal{P}}_*=0$, we get

$$\begin{aligned}-a(\widetilde{\nabla}_U\widetilde{R})(X,Y,Z,V)&=b\Big((\widetilde{\nabla}_U\widetilde{E})(Y,Z)g(X,V)-(\widetilde{\nabla}_U\widetilde{E})(X,Z)g(Y,V)\Big)\\&\quad-\frac{(\widetilde{\nabla}_U\widetilde{r})a}{n(n-1)}(g(Y,Z)g(X,V)-g(X,Z)g(Y,V)).\end{aligned} \tag{13}$$

Contracting Y and Z in (13) and using the condition (1) and (2), we have

$$\begin{aligned}-a(\widetilde{\nabla}_U\widetilde{S})(X,V)&=b\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})g(X,V)-(\widetilde{\nabla}_U\widetilde{E})(X,V)\\&\quad-\frac{(\widetilde{\nabla}_U\widetilde{r})a}{n}g(X,V).\end{aligned} \tag{14}$$

Taking $X=V=e_j$ in (14), we obtain

$$\begin{aligned}&b(n-1)\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})=0,\\&\operatorname{trace}(\widetilde{\nabla}_U\widetilde{E})=0,\qquad(\text{since } b\neq 0 \text{ and } n>2).\end{aligned} \tag{15}$$

□

Theorem 2. *Let (M,g) be an n-dimensional semi-Riemannian $\widetilde{\nabla}$-pseudo-projective symmetric manifold with $n>2$ and $b\neq 0$, then the equation (11) has maximum number of solution and consequently, $i_{\widetilde{\nabla}}=\frac{1}{2}n(n-1)$.*

Proof. Using (13) and (11), we find

$$\begin{aligned}0&=b((\widetilde{\nabla}_U\widetilde{E})(Y,Z)H(X,V)-(\widetilde{\nabla}_U\widetilde{E})(X,Z)H(Y,V)\\&\quad+(\widetilde{\nabla}_U\widetilde{E})(Y,V)H(X,Z)-(\widetilde{\nabla}_U\widetilde{E})(X,V)H(Y,Z))\\&\quad-\frac{a(\widetilde{\nabla}_U\widetilde{r})}{n(n-1)}(g(Z,Y)H(X,V)-g(Z,X)H(Y,V)\\&\quad+g(V,Y)H(X,Z)-g(V,X)H(Y,Z)).\end{aligned} \tag{16}$$

Taking $X=Z=e_i$ in (16) and using (15), we get

$$\begin{aligned}&b(H((\widetilde{\nabla}_U\widetilde{e})Y,V)-H((\widetilde{\nabla}_U\widetilde{e})V,Y)+(\widetilde{\nabla}_U\widetilde{E})(V,Y)\operatorname{trace}(H))\\&=\frac{a(\widetilde{\nabla}_U\widetilde{r})}{n(n-1)}(-nH(Y,V)+g(Y,V)\operatorname{trace}(H)).\end{aligned} \tag{17}$$

Interchanging V with Y in (17) and then subtracting the resulting equation from (17), we obtain

$$H((\widetilde{\nabla}_U\widetilde{e})Y,V)=H((\widetilde{\nabla}_U\widetilde{e})V,Y). \tag{18}$$

Using (18) in (17), we get

$$b(\widetilde{\nabla}_U\widetilde{E})(V,Y)=\frac{a(\widetilde{\nabla}_U\widetilde{r})}{n(n-1)}\left(g(Y,V)-\frac{n}{\operatorname{trace}(H)}H(Y,V)\right). \tag{19}$$

Now, interchanging X with Z, and Y with V in (16) and taking the sum of the resulting equation and (16) and using (19), we see that the equation (11) is satisfied identically. Thus the equation has the maximum number of solutions for a $\widetilde{\nabla}$-pseudo-projective symmetric semi-Riemannian manifold. Consequently, M admits the maximum number of linearly independent SQFI. So, the index of a $\widetilde{\nabla}$ -pseudo-projectively symmetric semi-Riemannian manifold is

$$i_{\widetilde{\nabla}}=\frac{1}{2}n(n-1).$$

□

Corollary 2. *If (M,g) is an n-dimensional semi-Riemannian $\widetilde{\nabla}$-projectively symmetric manifold, then the equation (11) has maximum number of solution and consequently, $i_{\widetilde{\nabla}}=\frac{1}{2}n(n-1)$.*

4 CONCLUSION

A semi-Riemannian manifold is said to be *decomposable* [17] (or locally reducible) if there always exists a local coordinate system (x^i) so that its metric takes the form

$$ds^2=\sum_{a,b=1}^{r}g_{ab}dx^adx^b+\sum_{\alpha,\beta=r+1}^{n}g_{\alpha\beta}dx^\alpha dx^\beta,$$

where g_{ab} are functions of $x^1,\ldots,x^r$ and $g_{\alpha\beta}$ are functions of $x^{r+1},\ldots,x^n$. A semi-Riemannian manifold is said to be *reducible* if it is isometric to the product of two or more semi-Riemannian manifolds; otherwise it is said to be *irreducible* [17]. A reducible semi-Riemannian manifold is always decomposable but the converse need not be true.

The concept of the index of a (semi-)Riemannian manifold gives a striking tool to decide the reducibility and decomposability of (semi-)Riemannian manifolds. For example, a Riemannian manifold is decomposable if and only if its index is greater than one [17]. Moreover, a complete Riemannian manifold is reducible if and only if its index is greater than one [17]. A second order $(0,2)$-symmetric parallel tensor is also known as a special Killing tensor of order two. Thus, a Riemannian manifold admits a special Killing tensor other than the Riemannian metric g if and only if the manifold is reducible [2], that is the index of the manifold is greater than 1. In 1951, Patterson [11] found a similar result for semi-Riemannian manifolds. In fact, he proved that a semi-Riemannian manifold (M, g) admitting a special Killing tensor K_{ij}, other than g, is reducible if the matrix (K_{ij}) has at least two distinct characteristic roots at every point of the manifold. In this case, the index of the manifold is again greater than 1.

By Theorem 2, we conclude that a $\tilde{\nabla}$-pseudo-projectively symmetric Riemannian manifold (where $\tilde{\nabla}$ is any Ricci symmetric metric connection, not necessarily Levi-Civita connection) is decomposable and it is reducible if the manifold is complete.

It is known that the maximum number of linearly independent Killing tensors of order 2 in a semi-Riemannian manifold (M^n, g) is $\frac{1}{12}n(n+1)^2(n+2)$, which is attained if and only if M is of constant curvature. The space of constant curvature and projectively flat space are identical classes. Therefore the maximum number of linearly independent Killing tensors of order 2 in a semi-Riemannian manifold (M^n, g) is $\frac{1}{12}n(n+1)^2(n+2)$, which is attained if and only if M is projectively flat. The maximum number of linearly independent Killing tensors in a 4-dimensional spacetime is 50 and this number is attained if and only if the spacetime is of constant curvature [5] or projectively flat. But spaces of constant curvature do not admit special quadratic first integrals. From Theorem 2, we also conclude that the maximum number of linearly independent special Killing tensors, that is, SQFI in a 4-dimensional spacetime is 6.

From the physical point of view Killing tensors are important because they provide quadratic first integrals of the geodesics. It is shown that [1] the special quadratic first integrals can be written as the sum of products of two linear first integrals only if the space admits a covariantly constant vector. Therefore special quadratic first integrals are useful in the analysis of the geodesics of given relativistic space-times possessing groups of motion of order less than or equal to 2.

The charged Kerr solution with or without cosmological constant admits a quadratic first integral which is irreducible provided the angular momentum parameter is not zero [20]. But this quadratic first integral is not special [1].

REFERENCES

[1] Collinson C.D. *Special quadratic first integrals of geodesics.* J. Phys. A: Gen. Phys. 1971, **4** (6), 756–760. doi:10.1088/0305-4470/4/6/002

[2] Eisenhart L.P. *Symmetric tensors of the second order whose first covariant derivatives are zero.* Trans. Amer. Math. Soc. 1923, **25** (2), 297–306. doi:10.2307/1989258

[3] Eisenhart L.P. Riemannian Geometry. Princeton University Press, Princeton, N. J., 1949.

[4] Golab S. *On semi-symmetric and quarter-symmetric linear connections.* Tensor (N.S.) 1975, **29**, 249–254.

[5] Hauser I., Malihot R.J. *Structural equations for Killing tensors of order two. I.* J. Math. Phys. 1975, **16**, 1625–1629. doi:10.1063/1.522407

[6] Hayden H.A. *Subspaces of a space with torsion.* Proc. London Math. Soc. (3) 1932, **34**, 27–50. doi:10.1112/plms/s2-34.1.27

[7] Levine J., Katzin G.H. *On the number of special quadratic first integrals in affinely connected and Riemannian spaces.* Tensor (N.S.) 1968, 113–118.

[8] Levine J., Katzin G. H. *Conformally flat spaces admitting special quadratic first integrals, I. Symmetric spaces.* Tensor (N.S.) 1968, **19**, 317–328.

[9] Levy H. *Symmetric tensors of the second order whose covariant derivatives vanish.* Ann. of Math. (2) 1925, **27** (2), 91–98. doi:10.2307/1967964

[10] O'Neill B. Semi-Riemannian geometry with applications to relativity. Academic Press, 1983.

[11] Patterson E.M. *On symmetric recurrent tensors of the second order.* Q. J. Math. 1952, **2**, 151–158.

[12] Prasad B. *A pseudo projective curvature tensor on a Riemannian manifold.* Bull. Calcutta Math. Soc. 2002, **94** (3), 163–166.

[13] Sharma R. *Second order parallel tensor in real and complex space forms.* Internat. J. Math. Math. Sci. 1989, **12** (4), 787–790.

[14] Stavre P. *On the index of a conformally symmetric Riemannian space.* An. Univ. Craiova Mat. Fiz.-Chim. 1981, **9**, 35–39.

[15] Stavre P. *On the S-concircular and S-conharmonic connections.* Tensor (N.S.) 1982, **38**, 103–107.

[16] Stavre P., Smaranda D. *On the index of a conformal symmetric Riemannian manifold with respect to the semisymmetric metric connection of K. Yano.* An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N. S.) 1982, **28** (1), 73–78.

[17] Thomas T.Y. *The decomposition of Riemann spaces in the large.* Monatsh. Math. Phys. 1939, **47**, 388–418.

[18] Tripathi M.M., Gupta P., Kim J.-S. *Index of quasi-conformally symmetric semi-Riemannian manifolds.* Int. Jour. Math. Math. Sci. 2012, **2012**, 461383.

[19] Tripathi M.M., Gupta P. *$\mathcal{T}$-curvature tensor on a semi-Riemannian manifolds.* J. Adv. Math. Stud. 2011, **4** (1), 117–129.

[20] Walker M., Penrose R. *On quadratic first integrals of the geodesic equations for type {22} spacetimes.* Comm. Math. Phys. 1970, **18**, 265–274. doi:10.1007/BF01649445

[21] Yano K. Concircular transformations. In: Concircular Geometry, 1. Proc. Imp. Acad. Tokyo., Tokyo, 1940. 16, 195–200.

[22] Yano K. *On semi-symmetric metric connection.* Rev. Roumaine Math. Pures Appl. 1970, **15**, 1579–1586.

[23] Yano K. *The Hayden connection and its applications.* Southeast Asian Bull. Math. 1982, **6** (2), 96–114.

[24] Yano K., Bochner S. Curvature and Betti numbers. In: Annals of Mathematics Studies, 32. Princeton Univ. Press, 1953.

Received 02.02.2015

Гупта П. *Індекс псевдопроективно симетричних напіврімановиx многовидів* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 57–65.

Досліджується індекс $\tilde{\nabla}$-псевдопроективно симетричних і зокрема $\tilde{\nabla}$-проективно симетричних напіврімановиx многовидів, де $\tilde{\nabla}$ — це симетричний метричний зв'язок Річчі.

Ключові слова і фрази: метричний зв'язок, псевдопроективний тензор кривизни, проективний тензор кривизни, напіврімановий многовид, індекс многовиду.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 66–71
doi:10.15330/cmp.7.1.66-71

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.66–71

Дільний В.М.[1], Війчук Т.І.[2]

ПРО ОДНУ РЕАЛІЗАЦІЮ ПРИНЦИПУ НЕВИЗНАЧЕНОСТІ

Отримано твердження про наслідування поведінки суми функцій на дійсній півосі кожним з доданків при певних умовах на ці функції та їх перетворення Лапласа.

Ключові слова і фрази: принцип невизначеності, простір Гарді, перетворення Фур'є.

[1] Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
[2] Ivan Franko State Pedagogical University, 24 Franka str., 82100, Drohobych, Ukraine
E-mail: dilnyi@ukr.net (Дільний В.М.), taras.viychuk@gmail.com (Війчук Т.І.)

Вступ

С. Мандельбройт (див. [1–3]) одержав теорему, яка стверджує, що коли функція f належить простору L^1 на одиничному колі, показник збіжності її спектру λ менший одиниці і для кожного $\rho > \lambda(1-\lambda)^{-1}$ виконується

$$\int\limits_0^{\varepsilon} |f(re^{it})|dt = O(\exp(-\varepsilon^{-\rho})), \quad \varepsilon \to 0,$$

то $f \equiv 0$. Інакше кажучи, якщо функція із простору L^1 має досить "негустий" спектр, то вона є тотожнім нулем. Це можна інтерпретувати також як твердження про те, що функція та її перетворення Фур'є не можуть одночасно бути дуже малими. Такого типу твердження відомі також як "принцип невизначеності в гармонічному аналізі" і отримали досить повний виклад в монографії [3]. Метою цього дослідження є отримання твердження про наслідування поведінки на дійсній півосі суми функцій кожним з доданків при певних умовах на ці функції та їх перетворення Лапласа (чи Фур'є).

1 Основний результат

Введемо позначення $D_{\alpha,\beta} = \{z : \text{Re}z < 0, \alpha < \text{Im}z < \beta\}$, $D^*_{\alpha,\beta} = \mathbb{C}\backslash\overline{D}_{\alpha,\beta}$, $\alpha < \beta$. Через $E^p[D_{\alpha,\beta}]$ та $E^p_*[D_{\alpha,\beta}]$, $1 \le p < +\infty$, позначимо простори функцій f, аналітичних відповідно в $D_{\alpha,\beta}$ і $D^*_{\alpha,\beta}$, для яких

$$\sup\left\{\int\limits_{\gamma} |f(z)|^p|dz|\right\} < +\infty,$$

УДК 517.5
2010 *Mathematics Subject Classification:* 30D10, 30H10, 35S30.

де супремум береться за всіма відрізками γ, що лежать відповідно в $D_{\alpha,\beta}$ і $D^*_{\alpha,\beta}$. Функції f з цих просторів мають майже скрізь [4] на ∂D_σ кутові граничні значення, які ми теж позначаємо через $f(z)$ і $f \in L^p[\partial D_{\alpha,\beta}]$. Також для фіксованого $\sigma > 0$ позначимо $D_1 = D_{-2\sigma,0}$, $D_1^* = D^*_{-2\sigma,0}$, $D_3 = D_{0,2\sigma}$, $D_3^* = D^*_{0,2\sigma}$.

Теорема 1. *Нехай*

$$q_j(w) = \frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty} \Omega_j(x)e^{-xw}dx, j \in \{1;3\}, \tag{1}$$

$$\lim_{x\to+\infty}\frac{\ln|\Omega_j(x)|}{x} = -\infty, \quad j \in \{1;3\}, \tag{2}$$

$$(\Omega_1(x) + \Omega_3(x))\, e^{ax\ln x} \in L^2(0;+\infty),\ a > 0, \tag{3}$$

причому q_1, q_3 є такими цілими функціями, що $q_1 \in E^2_*[D_1]$, $q_3 \in E^2_*[D_3]$. Тоді для кожної незростаючої функції $\varkappa : (0;+\infty) \to (-\infty;0)$, такої що $\varkappa(x) = O(x)$, $x \to +\infty$, при деякому $c \in \mathbb{R}$

$$\Omega_1(x) \in L^2(0;+\infty)e^{cz}e^{x\varkappa(x)}\exp\left\{\frac{a}{e}e^{-\frac{\varkappa(x)}{a}}\right\}, \tag{4}$$

$$\Omega_3(x) \in L^2(0;+\infty)e^{cz}e^{x\varkappa(x)}\exp\left\{\frac{a}{e}e^{-\frac{\varkappa(x)}{a}}\right\}.$$

Умови (2) і (3) зустрічаються в теорії циклічних функцій у вагових просторах Гарді [7]. Нижче показано, що умова (2) є істотною в теоремі 1. Нам не відомо, чи можна послабити інші умови на функції Ω_1, Ω_3 або їх перетворення Фур'є.

2 Доведення основного результату

Б. Винницький [5] розглянув простір $H^p_\sigma(\mathbb{C}_+)$, $\sigma \ge 0$, $1 \le p < +\infty$, аналітичних в $\mathbb{C}_+ := \{z : \text{Re}z > 0\}$ функцій, для яких

$$\sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}\left\{\int\limits_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|}dr\right\}^{1/p} < +\infty.$$

Функції з цього простору мають майже скрізь на $i\mathbb{R}$ кутові граничні значення, які також позначаємо через f і $f(iy)e^{-\sigma|y|} \in L^p(\mathbb{R})$. Для випадку $\sigma = 0$, як показав А.М. Седлецький в [11], $H^p_\sigma(\mathbb{C}_+)$ співпадає з простором Гарді в правій півплощині $H^p(\mathbb{C}_+)$.

Лема 1 ([5]). *Рівність*

$$G(z) = \frac{1}{i\sqrt{2\pi}}\int\limits_{\partial D_\sigma} g(w)e^{zw}dw$$

визначає бієкцію між просторами $H^2_\sigma(\mathbb{C}_+) \ni G$ і $E^2_*[D_\sigma] \ni g$. Також справджується двоїста формула

$$g(w) = \frac{1}{\sqrt{2\pi}}\int\limits_0^{+\infty} G(x)e^{-xw}dx.$$

Зауважимо, що в формулюванні теореми 1 умови $q_1 \in E_*^2[D_1]$, $q_3 \in E_*^2[D_3]$ можуть бути замінені еквівалентними їм внаслідок леми 1 умовами $\Omega_1(z)e^{-i\sigma z} \in H_\sigma^2(\mathbb{C}_+)$, $\Omega_3(z)e^{i\sigma z} \in H_\sigma^2(\mathbb{C}_+)$.

Доведення теореми 1. З леми 1 випливає, що для функцій, визначених рівністю (1), при виконанні умов $q_1 \in E_*^2[D_1]$, $q_3 \in E_*^2[D_3]$ справджуються рівності

$$\Omega_j(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_j} q_j(w)e^{zw}dw, \quad j \in \{1;3\}.$$

Покажемо, що функція q_1 є цілою. Справді, з умови (2) маємо, що $|\Omega_1(x)| = e^{x\eta(x)}$, причому $\eta(x) \to -\infty$ при $x \to -\infty$. Тому інтеграл в правій частині рівності (1) при $j = 1$ збігається абсолютно і рівномірно на кожному компакті з $\mathbb{C}$. Аналогічно з умови (2) маємо, що q_3 — ціла. Розглянемо функцію $\Omega_2 = -\Omega_1 - \Omega_3$. Тоді визначимо q_2 рівністю (1), поклавши в ній $j = 2$. Легко бачити, що на множині визначення функцій справедлива рівність $q_2 = -q_1 - q_3$ і тому q_2 — теж ціла. Звідси одержимо зображення

$$\Omega_1(z) = -\frac{1}{i\sqrt{2\pi}} \int_{\partial D_1} (q_2(w) + q_3(w))e^{zw}dw.$$

Але за означенням просторів маємо $E_*^2[D_3] \subset E^2[D_1]$, тому $q_3 \in E^2[D_1]$. З цього випливає, що $q_3(w)e^{wz} \in E^1[D_1]$ для кожного $z \in \mathbb{C}_+$, тому з [9] одержимо

$$\int_{\partial D_1} q_3(w)e^{zw}dw = 0, z \in \mathbb{C}_+.$$

Звідси маємо

$$\Omega_1(z) = -\frac{1}{i\sqrt{2\pi}} \int_{\partial D_1} q_2(w)e^{zw}dw.$$

Оскільки, як відмічено вище, q_2 є цілою функцією, то і функція $q_2(w)e^{zw}$ — ціла для кожного $z \in \mathbb{C}_+$. Тому вона аналітична в замиканнні кожного прямокутника $M_{\varkappa(x)} := \{w : w \in D_1, \operatorname{Re} z > \varkappa(x)\}$. Скориставшись інтегральною теоремою Коші для $M_{\varkappa(x)}$, маємо

$$\int_{\partial M_{\varkappa(x)}} q_2(w)e^{zw}dw = 0.$$

Тому для кожного $z \in \mathbb{C}_+$ справджується формула

$$\Omega_1(z) = -\frac{1}{i\sqrt{2\pi}} \int_{\partial(D_1 \setminus M_{\varkappa(x)})} q_2(w)e^{zw}dw, z \in \mathbb{C}_+.$$

Звідси для $x > 0$ маємо

$$|\Omega_1(x)| \le \frac{1}{\sqrt{2\pi}} \int_{\partial(D_1 \setminus M_{\varkappa(x)})} |q_2(w)|e^{xu}|dw| = \frac{1}{\sqrt{2\pi}}(I_1 + I_2 + I_3),$$

де $w = u + iv$, $k < 0$. З властивостей просторів $E_*^2[D_{\alpha,\beta}]$ випливає [5], що $q_1(u - 2i\sigma) \in L^2(-\infty;0)$ і $q_3(u - 2i\sigma) \in L^2(-\infty;0)$. Тому також $q_2(u - 2i\sigma) \in L^2(-\infty;0)$. Тоді з нерівності Шварца, врахувавши, що $q_2(u - 2i\sigma) \in L^2(-\infty;0)$, для $x > 0$ одержимо

$$I_1 = \int_{-\infty}^{\varkappa(x)} |q_2(u - 2i\sigma)|e^{xu}du \le \left(\int_{-\infty}^{\varkappa(x)} |q_2(u - 2i\sigma)|^2 du \cdot \int_{-\infty}^{\varkappa(x)} e^{2xu}du \right)^{\frac{1}{2}}$$
$$\le \left(\int_{-\infty}^{0} |q_2(u - 2i\sigma)|^2 du \cdot \frac{\exp(2x\varkappa(x))}{2x} \right)^{\frac{1}{2}} = \frac{c_1}{\sqrt{x}} \exp(x\varkappa(x)).$$

Аналогічно одержимо нерівність

$$I_3 = \int_{-\infty}^{\varkappa(x)} |q_2(u)|e^{xu}du \le \frac{c_2}{\sqrt{x}} \exp(x\varkappa(x)), \ x > 1.$$

Далі, скориставшись теоремою Фубіні, маємо

$$I_2 = \int_{-2\sigma}^{0} |q_2(\varkappa(x) + iv)|e^{x\varkappa(x)}dv = \exp(x\varkappa(x)) \int_{-2\sigma}^{0} \left| \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \Omega_2(t)e^{-t(\varkappa(x)+iv)}dt \right| dv$$
$$\le \frac{\exp(x\varkappa(x))}{\sqrt{2\pi}} \int_{-2\sigma}^{0} \int_{0}^{+\infty} \left| \Omega_2(t)e^{-t\varkappa(x)} \right| dtdv$$
$$\le \frac{\exp(x\varkappa(x))}{\sqrt{2\pi}} 2\sigma \left(\int_{0}^{+\infty} \left| \Omega_2(t)e^{at\ln t} \right|^2 dt \cdot \int_{0}^{+\infty} e^{-2at\ln t - 2t\varkappa(x)}dt \right)^{\frac{1}{2}}$$
$$\le c_3 \exp(x\varkappa(x)) \left(\exp\left\{ -\frac{\varkappa(x)}{2a} \right\} \exp\left\{ \frac{2a}{e} e^{-\frac{\varkappa(x)}{a}} \right\} \right)^{\frac{1}{2}}$$
$$= c_3 \exp\left\{ x\varkappa(x) - c_4\varkappa(x) + \frac{a}{e} e^{-\frac{\varkappa(x)}{2a}} \right\}.$$

Передостанній перехід випливає (див. [8, с. 323]) з асимптотичної рівності

$$\int_{0}^{+\infty} e^{-2at\ln t - 2t\varkappa(x)}dt = \sqrt{\frac{\pi}{ae}} \exp\left\{ -\frac{\varkappa(x)}{2a} \right\} \exp\left\{ \frac{2a}{e} e^{-\frac{\varkappa(x)}{a}} \right\} (1 + o(1)), \ x \to +\infty.$$

Тому для деяких невід'ємних сталих c_5, c_6 маємо

$$|\Omega_1(x)| \le c_5 e^{x\varkappa(x) - c_6\varkappa(x)} \exp\left\{ \frac{a}{e} e^{-\frac{\varkappa(x)}{a}} \right\}, \quad x > 0, \tag{5}$$

з чого випливає перша з доводжуваних формул, а друга доводиться аналогічно. □

3 Аналіз основної теореми

Цікавим для застосувань є випадок, коли останній множник в (4) дає незначний вклад в оцінку. Зокрема, коли $\varkappa(x) = \frac{2\sigma}{\pi} \ln x$ і $a = \frac{2\sigma}{\pi}$, можна одержати точніше твердження.

Теорема 2. *Нехай виконуються умови* (2) *та* (3), *причому* $q_1 \in E^2_*[D_1]$, $q_3 \in E^2_*[D_3]$. *Тоді знайдеться таке* $c \in \mathbb{R}$, *що*

$$\Omega_1(z)e^{-i\sigma z}e^{\frac{2\sigma}{\pi}z\ln z}e^{-cz} \in H^2(\mathbb{C}_+), \quad \Omega_3(z)e^{i\sigma z}e^{\frac{2\sigma}{\pi}z\ln z}e^{-cz} \in H^2(\mathbb{C}_+), \tag{6}$$

де $\ln z$ — *головне значення логарифма в* $\mathbb{C}_+$.

Доведення. Скористаємось теоремою типу Фрагмена-Ліндельофа (див. [9, 10]) для функції $\varphi_1(z) = \Omega_1(z)\exp\left\{-\frac{2\sigma}{\pi}z\ln z\right\}e^{-i\sigma z}e^{-c_5 z}$. Справді, з (5) одержимо $\varphi_1(x)e^{-\varepsilon x} \in L^2(0;+\infty)$, $\varepsilon > 0$. Оскільки також за умовами теореми і лемою 1 $\Omega_1(z)e^{-i\sigma z} \in H^2_\sigma(\mathbb{C}_+)$, то для кожного $\gamma \in (1;2]$

$$(\forall \varepsilon > 0): \sup_{|\varphi|<\frac{\pi}{2}} \left\{ \int_0^{+\infty} \left|\varphi_1(re^{i\varphi})\right|^2 \exp\{-\varepsilon r^\gamma\}\, dr \right\} < +\infty.$$

Легко бачити також, що $\varphi_1 \in L^2[i\mathbb{R}]$. З цього випливає, що $\varphi_1 \in H^2(\mathbb{C}_+)$. Цим доведено виконання першої з умов (6), а друга доводиться аналогічно. □

Зауваження 1. *Теорема 1 перестає справджуватися, якщо опустити в ній умову* (2).

Доведення. Розглянемо функцію $f(w) = \exp(-e^{-\frac{\pi}{2\sigma}w})e^w$. Очевидно, $f \in E^2[D_\sigma]$. Позначимо

$$F_j(z) = \frac{1}{\sqrt{2\pi}} \int_{l_j} f(w)e^{-wz}dw, \; j \in \{1;2;3\},$$

де l_1, l_2, l_3 — сторони ∂D_σ (відповідно півпряма, що лежить під дійсною віссю, відрізок $[-i\sigma, i\sigma]$ і півпряма, що лежить над дійсною віссю), орієнтація яких узгоджена з додатнім обходом ∂D_σ. За теоремою Пелі-Вінера функція F_2 належить простору Пелі-Вінера W^2_σ, тобто є цілою функцією, для якої виконується умова

$$\sup_{0<\varphi<2\pi} \left\{ \int_0^{+\infty} |F_2(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|}dr \right\} < +\infty.$$

Також позначимо $\Omega_j(z) = \exp\left(-\frac{2\sigma}{\pi}z\log z\right)F_j(z)$, $j \in \{1;2;3\}$. Тоді теж будемо мати $\Omega_2 = -\Omega_1 - \Omega_3$ і тому умова (3) виконується. Також справджуються зображення

$$\Omega_1(z) = \exp\left(-\frac{2\sigma}{\pi}z\log z\right) \int_{-\infty}^{0} \exp\left(-e^{-\frac{\pi}{2\sigma}(u-i\sigma)}\right) e^{u-i\sigma}e^{-(u-i\sigma)z}du,$$

$$\Omega_3(z) = \exp\left(-\frac{2\sigma}{\pi}z\log z\right) \int_{0}^{-\infty} \exp\left(-e^{-\frac{\pi}{2\sigma}(u+i\sigma)}\right) e^{u+i\sigma}e^{-(u+i\sigma)z}du,$$

тобто

$$\Omega_1(z) = \exp\left(-\frac{2\sigma}{\pi}z\log z\right) e^{-i\sigma}e^{i\sigma z} \int_{-\infty}^{0} \exp\left(-ie^{-\frac{\pi u}{2\sigma}}\right) e^{u}e^{-uz}du,$$

$$\Omega_3(z) = -\exp\left(-\frac{2\sigma}{\pi}z\log z\right) e^{i\sigma}e^{-i\sigma z} \int_{-\infty}^{0} \exp\left(ie^{-\frac{\pi u}{2\sigma}}\right) e^{u}e^{-uz}du.$$

В [6] показано, що для деякої сталої $c \in \mathbb{R}$ виконується $\Omega_1(z)e^{i\sigma z}e^{-cz} \in H^2_\sigma(\mathbb{C}_+)$, $\Omega_3(z)e^{-i\sigma z}e^{-cz} \in H^2_\sigma(\mathbb{C}_+)$. Тому з леми 1 випливає, що Ω_1 і Ω_3 задовільняють всім умови теореми 1 крім, можливо, умови (2). Але (див. [7]) для цієї пари функцій твердження теореми не справджується. Отже, для них умова (2) не виконується. □

4 Висновки

Нами одержано оцінки на дійсній осі для пари функцій при незначних обмеженнях (2) на кожну з них та жорстких обмеженнях на їх суму (3). Показано, що для часткового випадку $\varkappa(x) = \frac{2\sigma}{\pi}\ln x$, $a = \frac{2\sigma}{\pi}$ кожна з функцій в певному сенсі наслідує поведінку суми на дійсній осі. Також одержано оцінки вказаних функцій у правій півплощині. Вказано на істотність умови (2) теореми 1. Одержані результати можуть бути використані в теорії аналітичних функцій, зокрема при дослідженні просторів типу Гарді.

References

[1] Mandelbrojt S. Quasianalytical classes of functions. Gostekhizdat, Moscow-Leningrad, 1937. (in Russian)

[2] Levin B.Ja. Distribution of zeros of entire functions. Gostekhizdat, Moscow, 1956. (in Russian)

[3] Jöricke B., Khavin V.P. The uncertainty principle in harmonic analysis. In: Commutative harmonic analysis III, Itogi nauki i tekhniki, Ser. Sovrem. Probl. Mat., Fundam. Napravleniya, 72. VINITI, Moscow, 1991.

[4] Vynnyts'kyi B., Dil'nyi V. *On necessary conditions for the existence of solutions of one convolution-type equation.* Mat. Stud. 2001, **16** (1), 61–70. (in Ukrainian)

[5] Vinnitskii B. *On zeros of functions analytic in a half plane and completeness of systems of exponents.* Ukrainian Math. J. 1994, **46** (5), 514–532. doi:10.1007/BF01058515 (translation of Ukrain. Mat. Zh. 1994, **46** (5), 484–500. (in Ukrainian))

[6] Vinnitskii B. *Solutions of gomogeneous convolution equation in one class of functions analytical in a semistrip.* Mat. Stud. 1997, 7 (1), 41–52. (in Ukrainian)

[7] Dilnyi V. *On Cyclic Functions In Weighted Hardy Spaces.* Zh. Mat. Fiz. Anal. Geom. 2011, **7** (1), 19–33.

[8] Martirosian V. *On a Theorem of Djrbashian of the Phragmen-Lindelof Type.* Math. Nachr. 1989, **144** (1), 21–27. doi:10.1002/mana.19891440103

[9] Fedoryuk.M. Asymptotics, integrals and series. Nauka, Moscow, 1987. (in Russian)

[10] Vynnyts'kyi B., Dil'nyi V. *On solutions of homogeneous convolution equation generated by singularity.* Mat. Stud. 2003, **19** (2), 149–155.

[11] Sedletskii A.M. *An equivalent definition of spaces in the half-plane and some applications.* Math. Sb. USSR 1975, **25** (1), 69–76. doi:10.1070/SM1975v025n01ABEH002198

Надійшло 11.01.2014

Dilnyi V.M., Viychuk T.I. *A realization of the uncertainty principle.* Carpathian Math. Publ. 2015, **7** (1), 66–71.

We obtain the statement about the imitation behavior of the sum of functions on the real half-line by each of the summands under some conditions for these functions and their Laplace transforms.

Key words and phrases: the uncertainty principle, Hardy space, Fourier transform.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 72–77
doi:10.15330/cmp.7.1.72-77

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.72–77

Dmytryshyn R.I.

SOME PROPERTIES OF BRANCHED CONTINUED FRACTIONS OF SPECIAL FORM

The fact that the values of the approximates of the positive definite branched continued fraction of special form are all in a certain circle is established for the certain conditions. The uniform convergence of branched continued fraction of special form, which is a particular case of the mentioned fraction, in the some limited parabolic region is investigated.

Key words and phrases: branched continued fraction of special form.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: dmytryshynr@hotmail.com

Introduction

Several works are devoted to the establishment of different properties of branched continued fractions (BCF) of special form. For example, [1] is dedicated to the investigation of BCF with real positive and complex elements, [4] — to 1-periodic BCF of special form, [2] — to functional BCF with nonequivalent variables and BCF of special form with complex variables, [6] — to positive definite BCF of special form.

In this paper, using a representation of the approximants of BCF of special form (defined in [6]) through composition one- and two-dimensional fractional-linear maps, we have established that under certain conditions the values of the approximants of the positive definite BCF of special form

$$\Phi_0 + \cfrac{1}{b_{01}+z_{01}-\Phi_1+\mathop{D}\limits_{s=2}^{\infty}\dfrac{-a_{0s}^2}{b_{0s}+z_{0s}-\Phi_s}}, \quad \Phi_p = \cfrac{1}{b_{1p}+z_{1p}+\mathop{D}\limits_{r=2}^{\infty}\dfrac{-a_{rp}^2}{b_{rp}+z_{rp}}}, \quad p \ge 0, \qquad (1)$$

where a_{rs}, $r\ge 0, s\ge 0, r\ne 1, r+s\ge 2$, b_{rs}, $r\ge 0, s\ge 0, r+s\ge 1$, are complex numbers, z_{rs}, $r\ge 0, s\ge 0, r+s\ge 1$, are complex variables, are in a certain circle. Moreover, we investigated the converges uniformly of the BCF which is a particular case of the positive definite BCF of special form in the some limited parabolic domain.

1 Properties of BCF of special form

We show that under certain conditions the values of the approximants of the positive definite BCF of special form (1) are in a certain circle. For this we prove the following lemma.

УДК 517.524
2010 *Mathematics Subject Classification:* 11A55, 11J70, 30B70, 40A15.

Lemma 1. *Let*

$$t_{r,s-1}(w_{r+1,s-1}) = b_{r,s-1} + z_{r,s-1} - \frac{a_{r+1,s-1}^2}{w_{r+1,s-1}}, \quad t_{0s}(w_{1s}, w_{0,s+1}) = b_{0s} + z_{0s} - \frac{1}{w_{1s}} - \frac{a_{0,s+1}^2}{w_{0,s+1}},$$

where $r\ge 1,\ s\ge 1$, *and let*

$$\begin{aligned} &y_{01} = \operatorname{Im} z_{01} > 0, y_{1s} = \operatorname{Im} z_{1s} > 0, y_{rs} = \operatorname{Im} z_{rs} \ge 0, \quad r\ge 0, s\ge 0, r\ne 1, r+s\ge 2,\\ &\beta_{rs} = \operatorname{Im} b_{rs} \ge 0, \quad 0\le g_{rs}\le 1, r\ge 0, s\ge 0, r+s\ge 1,\\ &\alpha_{rs}^2 = (\operatorname{Im} a_{rs})^2 \le \beta_{rs}\beta_{r+\delta_{r0}-1,s-\delta_{r0}}(1-g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs}, \quad r\ge 0, s\ge 0, r\ne 1, r+s\ge 2, \end{aligned} \qquad (2)$$

where δ_{pq} *is the Kronecker's delta. If* $\operatorname{Im} w_{1,s+1} \ge \beta_{1,s+1}g_{1,s+1}$, $\operatorname{Im} w_{r-\delta_{r0}+1,s+\delta_{r0}} \ge \beta_{r-\delta_{r0}+1,s+\delta_{r0}}g_{r-\delta_{r0}+1,s+\delta_{r0}}$, *where* $r\ge 0,\ s\ge 0,\ r+s\ge 1$, *then*

$$\operatorname{Im} t_{rs}(w_{r+1,s}) \ge \beta_{rs}g_{rs} + y_{rs}, \quad r\ge 1,\ s\ge 0, \qquad (3)$$

$$\operatorname{Im} t_{0s}(w_{1s}, w_{0,s+1}) \ge \beta_{0s}g_{0s} + y_{0s}, \quad s\ge 1. \qquad (4)$$

Proof. The validity of the inequalities (3) follows from [5, Lemma 17.1]. We show that the inequalities (4) are valid. It is obvious that for arbitrary s, $s\ge 1$, provided that $\beta_{1s}g_{1s}+y_{1s}>0$ the image of half-plane $\operatorname{Im} w_{1s} \ge \beta_{1s}g_{1s}+y_{1s}$ under the transformation $t = w^{-1}$ is the circle

$$\left|\frac{1}{w_{1s}} + \frac{i}{2(\beta_{1s}g_{1s}+y_{1s})}\right| \le \frac{1}{2(\beta_{1s}g_{1s}+y_{1s})}.$$

Hence $\operatorname{Im}(1/w_{1s}) \le 0$. Let all $y_{0s}>0$. By the lemma for arbitrary s, $s\ge 1$, we have

$$\operatorname{Im} w_{0,s+1} \ge \beta_{0,s+1}g_{0,s+1}\frac{\beta_{0s}(1-g_{0s})}{\beta_{0s}(1-g_{0s})+y_{0s}} \ge \frac{\alpha_{0,s+1}^2}{\beta_{0s}(1-g_{0s})+y_{0s}}.$$

Therefore,

$$\left|w_{0,s+1} + \frac{i a_{0,s+1}^2}{2(\beta_{0s}(1-g_{0s})+y_{0s})}\right| \ge \frac{|a_{0,s+1}^2|}{2(\beta_{0s}(1-g_{0s})+y_{0s})}$$

or

$$\left|\frac{w_{0,s+1}}{a_{0,s+1}^2} + \frac{i}{2(\beta_{0s}(1-g_{0s})+y_{0s})}\right| \ge \frac{1}{2(\beta_{0s}(1-g_{0s})+y_{0s})}. \qquad (5)$$

The image of (5) under the transformation $w = 1/z$ is the half-plane

$$\operatorname{Im}\frac{a_{0,s+1}^2}{w_{0,s+1}} \le \beta_{0s}(1-g_{0s}) + y_{0s}.$$

Next, for arbitrary s, $s\ge 1$, we have

$$\operatorname{Im} t_{0s}(w_{1s}, w_{0,s+1}) = \beta_{0s} + y_{0s} - \operatorname{Im}\frac{1}{w_{1s}} - \operatorname{Im}\frac{a_{0,s+1}^2}{w_{0,s+1}} \ge \beta_{0s} + y_{0s} - \operatorname{Im}\frac{a_{0,s+1}^2}{w_{0,s+1}} \ge \beta_{0s}g_{0s}.$$

Going to the limit in the last inequality for $y_{0s}\to 0$, we obtain $\beta_{0s} - \operatorname{Im}\frac{a_{0,s+1}^2}{w_{0,s+1}} \ge \beta_{0s}g_{0s}$. Thus, $\operatorname{Im} t_{0s}(w_{1s}, w_{0,s+1}) = \beta_{0s} + y_{0s} - \operatorname{Im}\frac{1}{w_{1s}} - \operatorname{Im}\frac{a_{0,s+1}^2}{w_{0,s+1}} \ge \beta_{0s}g_{0s} + y_{0s}$, which had to be proved. □

Since the images of half-planes $\operatorname{Im} w_{10} \geq \beta_{10}g_{10} + y_{10}$ and $\operatorname{Im} w_{01} \geq \beta_{01}g_{01} + y_{01}$ under the transformation $t = t_0(w) = 1/w$ is respectively circles (nested or coincide)

$$\left|t + \frac{i}{2(\beta_{10}g_{10}+y_{10})}\right| \leq \frac{1}{2(\beta_{10}g_{10}+y_{10})}, \quad \left|t + \frac{i}{2(\beta_{01}g_{01}+y_{01})}\right| \leq \frac{1}{2(\beta_{01}g_{01}+y_{01})}$$

for $\beta_{10}g_{10} + y_{10} > 0$, $\beta_{01}g_{01} + y_{01} > 0$, then the image of transformation

$$t = t_0(w_{10}, w_{01}) = \frac{1}{w_{10}} + \frac{1}{w_{01}}$$

is the circle

$$\left|t + \frac{i(\beta_{10}g_{10}+y_{10}+\beta_{01}g_{01}+y_{01})}{2(\beta_{10}g_{10}+y_{10})(\beta_{01}g_{01}+y_{01})}\right| \leq \frac{\beta_{10}g_{10}+y_{10}+\beta_{01}g_{01}+y_{01}}{2(\beta_{10}g_{10}+y_{10})(\beta_{01}g_{01}+y_{01})},$$

which we denote by $K_0(\mathbf{z})$, where $\mathbf{z} = (z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, \dots)$ is infinite-dimensional vector.

For arbitrary n, $n \geq 1$, we define $K_n(\mathbf{z})$ as the map of the region

$$\operatorname{Im} w_{1n} \geq \beta_{1n}g_{1n}, \quad \operatorname{Im} w_{r-\delta_{r0}+1,s+\delta_{r0}} \geq \beta_{r-\delta_{r0}+1,s+\delta_{r0}}g_{r-\delta_{r0}+1,s+\delta_{r0}},$$

where $r \geq 0$, $s \geq 0$, $r+s=n$, under the transformation

$$T_n(w_{n+1,0}, w_{n1}, \dots, w_{0,n+1}) = \Phi_0^n + \frac{1}{b_{01}+z_{01}-\Phi_1^{n-1}} - \frac{a_{02}^2}{b_{02}+z_{02}-\Phi_2^{n-2}} - \frac{a_{03}^2}{b_{03}+z_{03}-\Phi_3^{n-3}} - \dots - \frac{a_{0,n-1}^2}{b_{0,n-1}+z_{0,n-1}-\Phi_{n-1}^1} - \cfrac{a_{0n}^2}{b_{0n}+z_{0n}-\cfrac{1}{w_{1n}}-\cfrac{a_{0,n+1}^2}{w_{0,n+1}}},$$

where

$$\Phi_k^{n-k} = \frac{1}{b_{1k}+z_{1k}} - \frac{a_{2k}^2}{b_{2k}+z_{2k}} - \frac{a_{3k}^2}{b_{3k}+z_{3k}} - \dots - \frac{a_{n-k,k}^2}{b_{n-k,k}+z_{n-k,k}} - \frac{a_{n-k+1,k}^2}{w_{n-k+1,k}}, \quad 0 \leq k \leq n-1.$$

Applying lemma 1 and taking into account (3) and (4), we have

$$K_0(\mathbf{z}) \supseteq K_1(\mathbf{z}) \supseteq K_2(\mathbf{z}) \supseteq \dots \tag{6}$$

Since (see [3, pp. 15–16]) $T_n(\underbrace{\infty, \infty, \dots, \infty}_{n+2}) = f_n(\mathbf{z})$, where $f_n(\mathbf{z})$ is the nth approximant of the BCF (1), then $f_n(\mathbf{z}) \in K_n(\mathbf{z})$, $n \geq 1$. Hence we prove the following theorem.

Theorem 1. *If the conditions (2) holds, where*

$$\beta_{rs} \geq 0, \quad \beta_{1s}g_{1s} + y_{1s} > 0, \quad \beta_{01}g_{01} + y_{01} > 0, \quad y_{rs} \geq 0, \quad r \geq 0,\ s \geq 0,\ r+s \geq 1,$$

then the approximants $f_n(\mathbf{z})$, $n \geq 1$, of the BCF (1) satisfy the inequalities

$$\operatorname{Im} f_n(\mathbf{z}) \leq 0, \quad |f_n(\mathbf{z})| \leq \frac{\beta_{10}g_{10}+y_{10}+\beta_{01}g_{01}+y_{01}}{(\beta_{10}g_{10}+y_{10})(\beta_{01}g_{01}+y_{01})}, \quad n \geq 1.$$

In [6] the notion of the nth denominator $B_n(\mathbf{z})$ of the approximant $f_n(\mathbf{z})$, $n \geq 1$, of BCF (1) is given. By arguments similar to the proof of the [3, Theorem 4.8], we can show that following theorem holds.

Theorem 2. *If the conditions (2) holds, where*

$$\beta_{rs}g_{rs} > 0, \quad r \geq 0,\ s \geq 0,\ r+s \geq 1,$$

then the denominators $B_n(\mathbf{z})$, $n \geq 1$, of the BCF (1) are different from zero for

$$\operatorname{Im} z_{rs} \geq 0, \quad r \geq 0,\ s \geq 0,\ r+s \geq 1.$$

2 POSITIVE DEFINITE BCF OF SPECIAL FORM AND THE PARABOLA THEOREM

Putting $z_{rs} = 0$, $b_{rs} = i$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$, in BCF (1), we obtain

$$\hat{\Phi}_0 + \cfrac{1}{i - \hat{\Phi}_1 - \mathop{\mathrm{D}}\limits_{s=2}^{\infty} \cfrac{a_{0s}^2}{i - \hat{\Phi}_s}}, \quad \hat{\Phi}_p = \cfrac{1}{i - \mathop{\mathrm{D}}\limits_{r=2}^{\infty} \cfrac{a_{rp}^2}{i}}, \quad p \geq 0. \tag{7}$$

Using the equivalent transformation [5, pp. 19–20], we put $\rho_0 = i$, $\rho_{rs} = i$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$, and BCF (7) reduce to

$$-i\tilde{\Phi}_0 + \cfrac{-i}{1 + \tilde{\Phi}_1 + \mathop{\mathrm{D}}\limits_{s=2}^{\infty} \cfrac{a_{0s}^2}{1 + \tilde{\Phi}_s}}, \quad \tilde{\Phi}_p = \cfrac{1}{1 + \mathop{\mathrm{D}}\limits_{r=2}^{\infty} \cfrac{a_{rp}^2}{1}}, \quad p \geq 0. \tag{8}$$

Next, putting $\rho_0 = 1/(1+\tilde{\Phi}_1)$, $\rho_{0s} = 1/(1+\tilde{\Phi}_{s+1})$, $s \geq 1$, we reduce the fraction (8) to the fraction with partial denominators equal to unity

$$-i\tilde{\Phi}_0 + \cfrac{\cfrac{-i}{1+\tilde{\Phi}_1}}{1 + \mathop{\mathrm{D}}\limits_{s=2}^{\infty} \cfrac{\cfrac{a_{0s}^2}{(1+\tilde{\Phi}_{s-1})(1+\tilde{\Phi}_s)}}{1}}. \tag{9}$$

Let

$$|a_{rs}^2| - \operatorname{Re} a_{rs}^2 \leq \frac{1}{2}, \quad |a_{rs}^2| \leq M, \quad M \geq 0, \quad r \geq 0,\ s \geq 0,\ r \neq 1,\ r+s \geq 2. \tag{10}$$

Then according to [5, Theorem 18.1] the continued fraction $\tilde{\Phi}_s$, $s \geq 0$, converges uniformly and according to [5, Theorem 14.3] the value of these fractions and of its approximants are in the domain $|z-1| \leq 1$, $z \neq 0$.

We take an arbitrary s, $s \geq 1$. The fraction $1/(1+\tilde{\Phi}_s)$ we write in the form $w = 1/(1+z)$. Hence $z = (1-w)/w$. Since $|z-1| \leq 1$, $z \neq 0$, then

$$\left|\frac{1-w}{w} - 1\right| \leq 1,\ w \neq 1 \quad \text{or} \quad |1-2w| \leq |w|,\ w \neq 1.$$

Let $w = x + iy$. Then

$$|1-2x-i2y| \leq |x+iy|, \quad (1-2x)^2 + 4y^2 \leq x^2 + y^2, \quad 3x^2 - 4x + 1 + 3y^2 \leq 0,$$

$$\left(x - \frac{2}{3}\right)^2 + y^2 \leq \frac{1}{9}, \quad \left|w - \frac{2}{3}\right| \leq \frac{1}{3}.$$

Thus, the value of the fractions $1/(1+\tilde{\Phi}_s)$, $s \geq 1$, and of its approximants are in the domain $|w - 2/3| \leq 1/3$, $w \neq 1$.

We put $1/(1+\tilde{\Phi}_s) = r_s e^{i\varphi_s}$, $s \geq 1$. Since the line $y = kx$ touches to the circle $3x^2 - 4x + 1 + 3y^2 = 0$ for $k = \pm 1/\sqrt{3}$, then $-\pi/6 \leq \varphi_s \leq \pi/6$, $s \geq 1$.

The following inequalities are valid for all $s \geq 2$

$$\left|\frac{1}{(1+\tilde{\Phi}_{s-1})(1+\tilde{\Phi}_s)}\right| \leq \cos^2 \frac{\varphi_{s-1}+\varphi_s}{2}, \quad -\frac{\pi}{6} \leq \varphi_{s-1}, \varphi_s \leq \frac{\pi}{6}. \tag{11}$$

Indeed, let $x = r\cos\varphi$, $y = r\sin\varphi$. Then the circle equation $3x^2 - 4x + 1 + 3y^2 = 0$ in polar coordinates we write in the form $3r^2 - 4r\cos\varphi + 1 = 0$ or

$$r = \frac{2\cos\varphi \pm \sqrt{4\cos^2\varphi - 3}}{3}, \quad -\frac{\pi}{6} \le \varphi \le \frac{\pi}{6}.$$

The inequalities (11) are equivalent to the inequalities

$$\frac{2\cos\varphi_{s-1} \pm \sqrt{4\cos^2\varphi_{s-1} - 3}}{3}\,\frac{2\cos\varphi_s \pm \sqrt{4\cos^2\varphi_s - 3}}{3} \le \cos^2\frac{\varphi_{s-1}+\varphi_s}{2}, \tag{12}$$

where $-\pi/6 \le \varphi_{s-1}, \varphi_s \le \pi/6$, $s \ge 2$.

To prove inequalities (12) is sufficient to show the validity of following inequalities

$$\frac{2\cos\varphi_{s-1} + \sqrt{4\cos^2\varphi_{s-1} - 3}}{3}\,\frac{2\cos\varphi_s + \sqrt{4\cos^2\varphi_s - 3}}{3} \le \cos^2\frac{\varphi_{s-1}+\varphi_s}{2}, \tag{13}$$

where $-\pi/6 \le \varphi_{s-1}, \varphi_s \le \pi/6$, $s \ge 2$.

Since $\sqrt{4\cos^2\varphi - 3} \le \cos\varphi$ for $-\pi/6 \le \varphi \le \pi/6$, then, estimating the top left side of the inequality (13), for any $-\pi/6 \le \varphi_{s-1}, \varphi_s \le \pi/6$, $s \ge 2$, we have

$$\cos\varphi_{s-1}\cos\varphi_s \le \cos^2\frac{\varphi_{s-1}+\varphi_s}{2} \quad \text{or} \quad \cos(\varphi_{s-1} - \varphi_s) \le 1.$$

That is, the inequalities (13) holds.

Applying relations (10) and (11), for any $s \ge 2$ we have

$$\left|\frac{a_{0s}^2}{(1+\tilde\Phi_{s-1})(1+\tilde\Phi_s)}\right| - \operatorname{Re}\frac{a_{0s}^2 e^{-i(\varphi_{s-1}+\varphi_s)}}{(1+\tilde\Phi_{s-1})(1+\tilde\Phi_s)} \le \frac{1}{2}\cos^2\frac{\varphi_{s-1}+\varphi_s}{2}, \quad -\frac{\pi}{6} \le \varphi_{s-1}, \varphi_s \le \frac{\pi}{6},$$

$$\left|\frac{a_{0s}^2}{(1+\tilde\Phi_{s-1})(1+\tilde\Phi_s)}\right| \le M, \quad M > 0.$$

According to [7, Theorem 4.40] the continued fraction

$$\cfrac{\cfrac{-i}{1+\tilde\Phi_1}}{1 + \mathop{\mathrm{D}}\limits_{s=2}^{\infty} \cfrac{\cfrac{a_{0s}^2}{(1+\tilde\Phi_{s-1})(1+\tilde\Phi_s)}}{1}}$$

converges uniformly. Hence, the fraction (9) converges uniformly too. From equivalence fractions (7) and (9) we conclude that BCF (7) converges uniformly, if the conditions (10) holds.

Hence, we have, if we change the notation, the following theorem holds.

Theorem 3. *BCF*

$$\Phi_0 + \cfrac{1}{1 + \Phi_1 + \mathop{\mathrm{D}}\limits_{s=2}^{\infty}\cfrac{a_{0s}}{1+\Phi_s}}, \quad \Phi_p = \cfrac{1}{1 + \mathop{\mathrm{D}}\limits_{r=2}^{\infty}\cfrac{a_{rp}}{1}}, \quad p \ge 0,$$

converges uniformly for all a_{rs} in the domain

$$P_M = \{z \in \mathbb{C} : |z| - \operatorname{Re} z \le 1/2, \quad |z| < M\}$$

for every constant $M > 0$.

REFERENCES

[1] Antomova T.M., Bodnar D.I. *Region convergence of branched continued fractions of special form.* Approx. Theor. and its Appl.: Pr. Inst. Matem. NAS Ukr. 2000, **31**, 19–32. (in Ukrainian)

[2] Baran O.E. *Some convergence regions of branched continued fractions of special form.* Carpathian Math. Publ. 2013, **5** (1), 4–13. doi: 10.15330/cmp.5.1.4-13 (in Ukrainian)

[3] Bodnar D.I. Branched continued fraction. Naukova Dumka, Kiev, 1986. (in Russian)

[4] Bodnar D.I., Bubnyak M.N. *On convergence of 1-periodic branched continued fraction of special form.* Math. Bull. Shevchenko Sci. Soc., **8**, 5–16. (in Ukrainian)

[5] Wall H.S. Analytic theory of continued fractions. Van Nostrand, New York, 1948.

[6] Dmytryshyn R.I. *Positive definite branched continued fractions of special form.* Carpathian Math. Publ. 2013, **5** (2), 225–230. doi: 10.15330/cmp.5.2.225-230

[7] Jones W.B., Thron W.J. Continued fractions: Analytic theory and applications. In: Encyclopedia of Mathematics and its Applications, 11. London, Amsterdam, Don Mills, Ontario, Sydney, Tokyo, 1980.

Received 05.03.2015

Revised 27.04.2015

Дмитришин Р.І. *Деякі властивості гіллястих ланцюгових дробів спеціального вигляду* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 72–77.

Встановлено, що за певних умов значення підхідних дробів додатно визначеного гіллястого ланцюгового дробу спеціального вигляду належать деякому кругу та досліджено рівномірну збіжність гіллястого ланцюгового дробу спеціального вигляду, який є частинним випадком такого дробу, в деякій обмеженій параболічній області.

Ключові слова і фрази: гіллястий ланцюговий дріб спеціального вигляду.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 78–82
doi:10.15330/cmp.7.1.78-82

http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2015, Т.7, №1, С.78–82

KOPACH M.I.[1], OBSHTA A.F.[2], SHUVAR B.A.[2]

TWO-SIDED INEQUALITIES WITH NONMONOTONE SUBLINEAR OPERATORS

Theorems on existence of solutions and their two-sided estimates for one class of nonlinear operator equations $x = Fx$ with nonmonotone operators are proved.

Key words and phrases: two-sided inequalities, extreme solutions, norm.

[1] Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
[2] Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine
E-mail: kopachm2009@gmail.com (Kopach M.I.), obshta2002@gmail.com (Obshta A.F.)

PROBLEM STATEMENT

While we studying in Banach space E the equation

$$x = Fx \tag{1}$$

with the nonlinear operator $F : E \to E$, in general, the following condition is often used

$$\lim_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} = \alpha. \tag{2}$$

For example, condition (2) for $\alpha = 0$ satisfies the integral equation

$$x(t) = f(t) + \int_D K(t,s)x^{\gamma}(s)ds, \quad 0 < \gamma < 1, \tag{3}$$

which was studied by M.A. Krasnoselsky [1] (see also references in [1]), B.Z. Vulykh [2], C.A. Stuart [3] and others. In particular, C.A. Stuart [3] uses the results obtained for equation (3) while investigating some boundary values problems for equations with partial derivatives.

In the present paper, we apply methods suggested in [4, §8] to the study of equation (1) with the operator F satisfying condition (2) under the assumption $0 < \alpha \leqslant 1$, which makes it possible to get some specification and generalization of respective results from [4, §8]. It also allows of applying the obtained results to the equation

$$x(t) = f(t) + \int_D K(t,s)x^{\gamma}(s)ds + \int_D K_1(t,s)x(s)ds, \quad 0 < \gamma \leqslant 1.$$

1 MAIN RESULTS AND THEIR EXPLANATION

Definition 1. *The norm of the pair of elements from E is $\|y,z\|$, satisfying the following conditions: a) if $y,z \in E$, the inequalities are $\|y\| \leqslant \|y,z\|$, $\|z\| \leqslant \|y,z\|$; b) the norm $\|y,z\|$ is monotone when introducing into $E \times E$ semiorderedness of pairs (y,z) elements from E, generated in this or that way of semiorderedness in E.*

УДК 517.968.4
2010 *Mathematics Subject Classification:* 41A65, 30B70.

For example, if the norm $\|y,z\|$ is introduced with the help of one of the formulas

$$\|y,z\| = \max\{\|y\|,\|z\|\}, \quad \|y,z\| = (\|y\|^p + \|z\|^p)^{\frac{1}{p}}, \quad p \geqslant 1, \tag{4}$$

and the semiorderedness of pairs (y,z) is defined as $(y,z) \leqslant (u,v)$ for $y \leqslant u$, $z \geqslant v$ or $(y,z) \leqslant (u,v)$ for $y \leqslant u$, $z \leqslant v$, then conditions a), b) are satisfied. With this semiorderedness, the space $E \times E$ is a fully regular space if E has this property.

Theorem 1. *Suppose that: 1) there are nondecreasing with respect to y and nonincreasing with respect to z continuous operators $T_1(y,z), T_2(y,z) : E \times E \to E$, such that*

$$T_1(x,x) = T_2(x,x) = Fx, \quad x \in E; \tag{5}$$

2) there exists $M > 0$, and from the inequality $\|y,z\| > M$ it follows that $\|T_1(y,z), T_2(z,y)\| \leqslant \|y,z\|$; 3) if $y,z \in E$, then $T_1(y,z) \leqslant T_2(y,z)$; 4) there are elements $u,v \in E$, for which $u \leqslant T_1(u,v)$, $v \geqslant T_2(v,u)$; 5) simultaneous equations

$$y = T_1(y,z), \quad z = T_2(z,y) \tag{6}$$

have no more than one solution; 6) equation (1) *has at least one solution. Then the unique solution $x^* \in E$ of equation* (1) *satisfies the estimates*

$$y_n \leqslant y_{n+1} \leqslant x^* \leqslant z_{n+1} \leqslant z_n, \quad n = 0,1,\ldots, \tag{7}$$

where the sequences $\{y_n\}$, $\{z_n\}$ are built with the help of

$$y_0 = u, \quad z_0 = v, \quad y_{n+1} = T_1(y_n,z_n), \quad z_{n+1} = T_2(z_n,y_n), \quad n = 0,1,\ldots \tag{8}$$

At that the sequences $\{y_n\}$, $\{z_n\}$ converge in E to x^* by the norm.

Proof. Let us prove the inequalities

$$y_0 \leqslant y_1 \leqslant \ldots \leqslant y_n \leqslant y_{n+1} \leqslant \ldots, \quad z_0 \geqslant z_1 \geqslant \ldots \geqslant z_n \geqslant z_{n+1} \geqslant \ldots \tag{9}$$

If $n = 0$ from conditions 4) and (8) we obtain $y_0 = u \leqslant T_1(y_0,z_0) = y_1, z_0 = v \geqslant T_2(z_0,y_0) = z_1$. Assuming that $y_{n-1} \leqslant y_n$, $z_{n-1} \geqslant z_n$, based on (8) and 1) we get

$$y_{n+1} = T_1(y_n,z_n) \geqslant T_1(y_{n-1},z_{n-1}) = y_n, \quad z_{n+1} = T_2(z_n,y_n) \leqslant T_2(z_{n-1},y_{n-1}) = z_n.$$

By induction, we come to a conclusion that the inequalities (9) are valid for any $n \in \mathbb{N}$.

Let us make sure that the sequences $\{y_n\}$, $\{z_n\}$ are limited by the norm. If starting from some number $n = N$, all the members of the sequence $\{(y_n,z_n)\}$ satisfied the inequality

$$\|(y_n,z_n)\| \leqslant M, \tag{10}$$

then the sequence $\{(y_n,z_n)\}$ would be limited by the norm, and the sequences $\{y_n\}$, $\{z_n\}$ would be limited by the norm. Assuming that for any $N > 0$ we have $n > N$ so that

$$\|y_n,z_n\| > M, \tag{11}$$

let us consider two mutually exclusive cases. Let in the first one exist no more than a finite number of the members of the sequence $\{(y_n,z_n)\}$, for which $\|y_n,z_n\| \leqslant M$. Then starting from some number $n = N$ inequality (11) holds. In virtue of (8) and condition 2) we obtain

$$\|y_{N+1},z_{N+1}\| = \|T_1(y_N,z_N), T_2(z_N,y_N)\| \leqslant \|y_n,z_n\|.$$

Assuming that $\|y_{N+k}, z_{N+k}\| \leqslant \|y_{N+k-1}, z_{N+k-1}\|$, we shall similarly find that

$$\|y_{N+k+1}, z_{N+k+1}\| = \|T_1(y_{N+k}, z_{N+k}), T_2(z_{N+k}, y_{N+k})\| \leqslant \|y_{N+k}, z_{N+k}\|.$$

By induction, we come to a conclusion that the sequence $\{(y_n, z_n)\}$ and the sequences $\{y_n\}$, $\{z_n\}$ are limited by the norm. Suppose that the sequence $\{(y_n, z_n)\}$ has an infinite number of members for which inequality (10) holds as well as an infinite number of members for which inequality (11) holds. It means that this property pertains to the sequence $\{y_n\}$, $\{z_n\}$. Let us choose arbitrary n_1 and n_2, $n_1 < n_2$, for which, for example, $\|y_{n_1}\| \leqslant M$, $\|y_{n_2}\| \leqslant M$. Let us have n_3 so that $n_1 < n_3 < n_2$ and $\|y_{n_3}\| > M$, from (9) we obtain $y_{n_1} \leqslant y_{n_3} \leqslant y_{n_2}$. Based on Lemma 8.1 [4, p. 37] we get $\|y_{n_3}\| \leqslant \|y_{n_1}\| + \|y_{n_2}\| \leqslant 2M$. This proves that the sequence $\{y_n\}$ is limited by the norm. It is similarly proved that the sequence $\{z_n\}$ is also limited by the norm. For the fully regular ordered space E, the monotonely nondecreasing sequence $\{y_n\}$ and the monotonely nonincreasing sequence $\{z_n\}$, which are limited by the norm, have limits y^* and z^*, $y^*, z^* \in E$, which are components of the solution of system (6). The solution $x^* \in E$ of equation (1) and equality (5) mean that (x^*, x^*) is the solution of system (6). Since system (6) has a unique solution, then $y^* = z^* = x^*$. The proof of the theorem is complete. □

Theorem 2. *Suppose that: 1) there are nondecreasing with respect to y, nonincreasing with respect to z continuous operators $T_1(y, z), T_2(y, z) : E \times E \to E$, such that*

$$T_1(x, x) \leqslant Fx \leqslant T_2(x, x), \quad x \in E; \tag{12}$$

2) conditions 2)–6) of Theorem 1 are satisfied. Then for any solution $x^* \in E$ of equality (1) *we have inequalities* (7)*, where sequences $\{y_n\}$, $\{z_n\}$ are built with the help of formulae* (8)*. Besides, the sequences $\{y_n\}$, $\{z_n\}$ converge to the components y^*, z^* of the solution (y^*, z^*) of system* (6) *and the estimates $u \leqslant y^* \leqslant x^* \leqslant z^* \leqslant v$ are valid.*

Proof. Let us build an iteration process with the help of

$$\varphi_0 = \psi_0 = x^*, \quad \varphi_{n+1} = T_1(\varphi_n, \psi_n), \quad \psi_{n+1} = T_2(\psi_n, \varphi_n), \quad n = 0, 1, \ldots,$$

where x^* is the solution of equation (1). From inequality (12), nondecreasing with respect to y and nonincreasing with respect to z properties of operators $T_1(y, z)$, $T_2(y, z)$ are observed by

$$\varphi_0 \geqslant \varphi_1 \geqslant \ldots \geqslant \varphi_n \geqslant \varphi_{n+1} \geqslant \ldots, \quad \psi_0 \leqslant \psi_1 \leqslant \ldots \leqslant \psi_n \leqslant \psi_{n+1} \leqslant \ldots, \quad n = 0, 1, \ldots$$

As in the proof of Theorem 1, we find that the sequence $\{\varphi_n\}$ converges to its limit φ^* without its monotone increase, and the sequence $\{\psi_n\}$ converges to its limit ψ^* without its monotone decrease. At that (φ^*, ψ^*) is the solution of system (6) and $\varphi^* \leqslant x^* \leqslant \psi^*$. Besides, for the sequences $\{y_n\}$, $\{z_n\}$, built with the help of formulae (8), we can fully repeat relevant considerations in the proof of Theorem 1 and reach the same conclusions concerning y^*, z^* as a component of the solution (y^*, z^*) of system (6) and about inequalities (9). The solution of system (6) being unique, it makes us possible to state that $\varphi^* = y^*$, $\psi^* = z^*$.

The proof of the theorem is complete. □

Theorem 3. *Suppose that: 1) condition 1) of Theorem 2 is satisfied; 2) there are linear positive relative to $w \in E$, nondecreasing with respect y, nonincreasing with respect to z operators $A_1(y, z)w$, $A_2(y, z)w$, for which if $x, y, z \in E$ the following inequalities hold*

$$-A_1(z, y)(z - y) \leqslant T_1(z, x) - T_1(y, x), \quad T_2(x, z) - T_2(x, y) \leqslant A_2(z, y)(z - y);$$

3) there is $M > 0$, so that from the inequality $\|z, y\| \geqslant M$ follows

$$\|T_1(y, z) - (A_1(z, y) + A_2(z, y))(z - y), T_2(y, z) + (A_1(z, y) + A_2(z, y))(z - y)\| \leqslant \|z, y\|;$$

4) there are $u, v \in E$, such that

$$u \leqslant -(A_1(v, u) + A_2(v, u))(v - u) + T_1(u, v), \quad v \geqslant (A_1(v, u) + A_2(v, u))(v - u) + T_2(v, u);$$

5) simultaneous equations

$$\begin{aligned} y &= -(A_1(z, y) + A_2(z, y))(z - y) + T_1(y, z), \\ z &= (A_1(z, y) + A_2(z, y))(z - y) + T_2(z, y) \end{aligned} \tag{13}$$

have in $E \times E$ no more than one solution. Then if there is a solution of equation (1)*, it is unique and the sequences $\{y_n\}$, $\{z_n\}$ converge to it without increasing and decreasing respectively. These sequences are built with the help of formulae*

$$\begin{aligned} y_{n+1} &= -(A_1(z_n, y_n) + A_2(z_n, y_n))(z_n - y_n) + T_1(y_n, z_n), \\ z_{n+1} &= (A_1(z_n, y_n) + A_2(z_n, y_n))(z_n - y_n) + T_2(z_n, y_n), \quad n = 0, 1, \ldots \end{aligned}$$

if $y_0 = u$, $z_0 = v$. Besides, there are estimates (7).

Proof of the sequences $\{y_n\}$, $\{z_n\}$ being monotone and limited by the norm in fact follows along the lines of respective considerations from the proof of Theorem 1. That's why $y_n \uparrow y^*$, $z_n \downarrow z^*$ $(y^*, z^* \in E)$, and (y^*, z^*) is solution of system (13). If x^* is the solution of equation (1), then (x^*, x^*) is the solution of system (13), and this system can have no more than one solution. The proof of the theorem is complete. □

2 APPLYING LIMITED ELEMENTS TO EQUATIONS IN KN-SPACES

Theorem 4. *Suppose that: 1) E is KN-space of limited elements and in $E \times E$ the norm is defined with the help of the first formula from* (4)*; 2) condition 2) of Theorem 1 and condition 1) of Theorem 2 are satisfied; 3) if $y \leqslant z$ $(y, z \in E)$, then $T_1(y, z) \leqslant T_2(z, y)$. Then there is extreme (see, e.g., [4, p. 22]) in $E \times E$ solution (y^*, z^*) of system* (6)*, the components of which belong to some segment $[-a, a] \subset E$, and for any solution $x^* \in E$ of equation* (1) *we have*

$$-a \leqslant x^* \leqslant a, \quad y^* \leqslant x^* \leqslant z^*. \tag{14}$$

Proof. If we replace condition 2) of Theorem 1 by the condition: if $\|y, z\| \geqslant M$ we have

$$\|T_1(y, z), T_2(z, y)\| < \|y, z\|, \tag{15}$$

then any solution (y, z), $y, z \in E$ of system (6) is within $D = \{(y, z) | \|y, z\| < M, \quad y, z \in E\}$. If for some solution (y, z) $(y, z \in E)$ of system (6) we have $\|y, z\| \geqslant M$, then from (6) and (15) we obtain $\|y, z\| = \|T_1(y, z), T_2(z, y)\| < \|y, z\|$, which is impossible. It allows us to draw a conclusion that any solution of system (6) belongs to the segment $[-a; a]$. If e is a unit of the space E of limited elements, it follows from what has been said that

$$|y| \leqslant \|y\|e \leqslant \|y, z\|e \leqslant Me, \quad |z| \leqslant \|z\|e \leqslant \|y, z\|e \leqslant Me.$$

Let us denote $Me = a$. Considering obvious inequality $-a \leqslant a$, inequality (15) and determination of domain D, we shall have

$$|T_1(-a, a)| \leqslant \|T_1(-a, a), T_2(a, -a)\|e \leqslant \| -a, a\|e = Me = a,$$

$$|T_2(a,-a)| \leqslant \|T_1(-a,a), T_2(a,-a)\|e \leqslant \|-a,a\|e = Me = a.$$

This implies that $-a \leqslant T_1(-a,a)$, $a \geqslant T_2(a,-a)$. To prove existence of extreme solution (y^*, z^*) of system (6) on the segment $[-a,a]$, it is enough to use iterations (8) setting $u = -a$, $v = a$ in them. As any solution of system (6) has components belonging to the segment $[-a,a]$, we draw a conclusion that (y^*, z^*) is extreme in $E \times E$ solution of system (6). The proof of the theorem is complete. □

Theorem 5. *Suppose that condition 1) of Theorem 4, condition 1) of Theorem 2 and condition 2) of Theorem 3 are satisfied. Then there is an extreme in* $E \times E$ *solution* (y^*, z^*) *of system* (13)*, the components of which belong to some segment* $[-a,a] \subset E$*, and for any solution* $x^* \in E$ *of equation* (1)*, there are estimates* (14).

The proof differs from the proof of Theorem 4 unessentially.

REMARK

If $T_1(y,z)$, $T_2(y,z)$ are fully continuous operators, then for the solution of equation (1) to exist, it is enough to satisfy condition 2) of Theorem 1. In this case, the operator generated by the right member of (6) will turn some sphere S of the radius M from $E \times E$ into compact in $E \times E$ set D_1. Let us choose the number $M_1 > M$ so high that the sphere $S_1 \subset E \times E$ contains the sphere S, as well as the compact, and therefore limited, set D_1. Thus, it turns out that the operator generated by the right member of (6), turns the sphere S into itself. Therefore, let us apply the Schauder principle.

Obtaining results supplement and specify results [5, §21] (see also references in [5]).

REFERENCES

[1] Krasnoselsky M.A. *Some problems of nonlinear analysis*. Usp. Math Nauk 1954, **9** (3), 57–114. (in Russian)

[2] Vulykh B.Z. *Introduction to semiordered spaces*. Fismatgiz, 1961. (in Russian)

[3] Stuart C.A. *Integral equations with geareasing nonlinearitres and applications*. J. Differential Equations 1975, **18** (1), 202–217. doi:10.1016/0022-0396(75)90090-X

[4] Kurpel N.S., Shuvar B.A. *Twosided operator inequalities and their application*. Naukova Dumka, Kyiv, 1980. (in Russian)

[5] Shuvar B.A., Kopach M.I., Mentynskyy S.M., Obshta A.F. Two-sided approximate methods. VDV CIT, Ivano-Frankivs'k, 2007. (in Ukrainian)

Received 22.12.2014

Revised 18.05.2015

Копач М.І., Обшта А.Ф., Шувар Б.А. *Двосторонні нерівності з немонотонними підлінійними операторами* // Карпатські матем. публ. — 2015. — Т.7, №1. — С. 78–82.

Встановлено теореми про існування розв'язків та їх двосторонніх оцінок для одного класу нелінійних операторних рівнянь вигляду $x = Fx$ з немонотонними операторами.

Ключові слова і фрази: двосторонні нерівності, крайні розв'язки, норма.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 83–90
doi:10.15330/cmp.7.1.83-90

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.83–90

KRASNIQI XH.Z.

ON A NECESSARY CONDITION FOR $L^p(0 < p < 1)$−CONVERGENCE (UPPER BOUNDEDNESS) OF TRIGONOMETRIC SERIES

In this paper we prove that the condition $\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{\lambda_k(p)}{(|n-k|+1)^{2-p}} = o(1)\ (= O(1))$, is a necessary condition for the $L^p(0 < p < 1)$−convergence (upper boundedness) of a trigonometric series. Precisely, the results extend some results of A. S. Belov [1].

Key words and phrases: trigonometric series, L^p−convergence, Hardy-Littlewood's inequality, Bernstein-Zygmund inequalities.

University of Prishtina, Mother Teresa str., 10 000 Prishtine, Prishtina, Kosovo
E-mail: xheki00@hotmail.com

1 INTRODUCTION AND PRELIMINARIES

Let

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx\right) \tag{1}$$

be a trigonometric series in the complex or real form respectively, and we use the following standard notations for all $n \geq 0$

$$\begin{aligned}
a_n &= c_n + c_{-n}, \\
b_n &= (c_n - c_{-n})i, \\
\lambda_n(p) &= \sqrt{2\left(|c_n|^{2p} + |c_{-n}|^{2p}\right)}, \\
r_n &= \sqrt{|a_n|^2 + |b_n|^2} = \sqrt{2(|c_n|^2 + |c_{-n}|^2)}, \\
A_n(x) &= c_n e^{inx} + c_{-n} e^{-inx} = a_n \cos nx + b_n \sin nx, \\
S_n(x) &= c_0 + \sum_{k=1}^{n} A_k(x), \\
\sigma_n(x) &= \frac{1}{n+1} \sum_{k=1}^{n} S_k(x), \\
\widetilde{S}_n(x) &= \sum_{k=1}^{n} (a_k \sin kx - b_k \cos kx) = -i \sum_{k=0}^{n} \left(c_k e^{ikx} - c_{-k} e^{-ikx}\right).
\end{aligned}$$

The square brackets denote the integer part of a number.

УДК 517.5
2010 *Mathematics Subject Classification:* 42A16, 42A20.

For $f \in L_{2\pi}$ the L^1-metric is defined by the equality

$$\|f\|_{L^1} = \|f\| = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|dx.$$

Regarding to the series (1) Belov [1] has proved a necessary condition, expressed in terms of its coefficients, for the L^1-convergence or L^1-boundedness of its partial sums proving the following statement.

Theorem 1. *If $n \geq 2$ is an integer, then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{r_k}{|n-k|+1} \leq 100 \max_{m=[n/2]-1,\dots,2n} \|\sigma_m - S_m\|.$$

In particular:

1. *If*

$$\|\sigma_m - S_m\| = o(1)\ (= O(1)), \tag{2}$$

then

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{r_k}{|n-k|+1} = o(1)\ (= O(1) \text{ respectively}). \tag{3}$$

2. *Assume that series (1) converges (possesses bounded partial sums) in the L^1-metric, then condition (3) holds.*

Also the author has considered the cosine and sine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{4}$$

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{5}$$

where for the series (4) or (5) the coefficients a_n are the same as in the trigonometric series (1) except for coefficients of series (5) which are denoted a_n instead of b_n, and the following corollary has been proved by him.

Corollary 1. *1. Assume that series* (4) *or* (5) *satisfies condition* (2), *then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{|a_k|}{|n-k|+1} = o(1)\ \ (O(1) \text{ respectively}). \tag{6}$$

2. Assume that series (4) *or* (5) *converges (possesses bounded partial sums) in the L^1-metric, then condition* (6) *holds.*

For $f \in L^p_{2\pi}$, $0 < p < 1$, the L^p-metric is defined by the equality

$$\|f\|_{L^p} = \|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}.$$

Of course $\|\cdot\|_p$ for $0 < p < 1$ is not a norm, it does not satisfy the triangle property, and it is known as quasi-norm.

The above statements, for $r-$th derivative of the series (1.1), has been generalized by present author in [2]. But nothing seems to be done so far concerning L^p-convergence $(0 < p < 1)$ of the series (1) in the direction as Belov did in [1]. Therefore, our main goal in this paper is studying of L^p-convergence of the series (1) for $0 < p < 1$.

Our main tools in proving the main results are Bernstein-Zygmund's inequality and Hardy-Littelwood's theorem in the spaces L^p $(0 < p < 1)$, and H^p $(0 < p < 1)$, respectively.

Lemma 1 ([3] or [5]). *Let $T_n(x)$ be a trigonometrical polynomial of order n and $0 < p < 1$. Then the inequality $\|T_n'\|_p \leq C_p n \|T_n\|_p$ holds true.*

Lemma 2 ([4]). *If $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$, $|z| < 1$ and $\varphi \in H^p, 0 < p < 1$, then*

$$\sum_{k=0}^{\infty} (k+1)^{p-2} |c_k|^p \leq C_p \|\varphi\|_p^p.$$

Throughout in this paper C_p denotes a positive constant that depends only on p, not necessarily the same at each occurrences.

2 MAIN RESULTS

We begin with the following helpful statements.

Lemma 3. *For every $m \in \mathbb{N}$ and $0 < p < 1$, we have*

$$\min\left\{\left\|\sum_{k=0}^{m} c_k e^{ikx}\right\|_p^p, \left\|\sum_{k=0}^{m} c_k e^{-ikx}\right\|_p^p\right\}$$
$$\geq \frac{1}{C_p} \max\left\{\sum_{k=0}^{m} (k+1)^{p-2}|c_k|^p, \sum_{k=0}^{m} (m-k+1)^{p-2}|c_k|^p\right\}.$$

Proof. The proof of this lemma is an immediate result of the Lemma 2. Indeed, we have that

$$\begin{aligned}
\left\|\sum_{k=0}^{m} c_k e^{ikx}\right\|_p^p &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=0}^{m} c_k e^{ikx}\right|^p dx = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=0}^{m} \bar{c}_k e^{-ikx}\right|^p dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|e^{imx}\sum_{k=0}^{m} \bar{c}_k e^{-ikx}\right|^p dx = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=0}^{m} \bar{c}_k e^{i(m-k)x}\right|^p dx \\
&= \left\|\sum_{k=0}^{m} \bar{c}_k e^{i(m-k)x}\right\|_p^p \geq \frac{1}{C_p}\sum_{k=0}^{m} (m-k+1)^{p-2}|c_k|^p.
\end{aligned}$$

The inequalities for $\sum_{k=0}^{m} c_k e^{-ikx}$ one can prove in the same way in view of equality

$$\left\|\sum_{k=0}^{m} c_k e^{-ikx}\right\|_p = \left\|\sum_{k=0}^{m} \bar{c}_k e^{ikx}\right\|_p.$$

□

Lemma 4. *Given an arbitrary trigonometric series (1) and arbitrary natural numbers n and N such that $n \le N \le 2n+1$. Then for $0<p<1$ the following estimates hold:*

$$\max_{k=n,\dots,N} \|\widetilde{S}_{n-1} - \widetilde{S}_k\|_p \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p, \tag{7}$$

$$\max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_j e^{ijx} \right) \right\|_p \le C_p \max_{m=n,\dots,N} \|S_m - S_{n-1}\|_p, \tag{8}$$

$$\max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_{-j} e^{-ijx} \right) \right\|_p \le C_p \max_{m=n,\dots,N} \|S_m - S_{n-1}\|_p, \tag{9}$$

$$\max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p \le C_p \max_{k=n-1,\dots,N} \|S_k - \sigma_k\|_p, \tag{10}$$

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p, \tag{11}$$

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(N+1-k)^{2-p}} \right)^{1/p} \le C_p \|S_N - S_{n-1}\|_p, \tag{12}$$

where C_p is a positive constant depending only on p.

Proof. (7). Let m, n be two natural numbers such that $m \ge n$. Using the equality

$$\widetilde{S}_{n-1}(x) - \widetilde{S}_m(x) = \frac{1}{m}\left(S'_m(x) - S'_{n-1}(x)\right) + \sum_{k=n}^{m-1} \frac{1}{k(k+1)}\left(S'_k(x) - S'_{n-1}(x)\right),$$

Lemma 1 and well-known inequalities

$$|a+b|^\beta \le \begin{cases} |a|^\beta + |b|^\beta, & \text{if } 0<\beta<1 \\ 2^\beta\left(|a|^\beta + |b|^\beta\right), & \text{if } \beta \ge 1 \end{cases}$$

we have that $\|S'_k - S'_{n-1}\|_p \le C_p k \|S_k - S_{n-1}\|_p$, and

$$\begin{aligned} \|\widetilde{S}_{n-1} - \widetilde{S}_m\|_p &\le 2^{\frac{1}{p}} C_p \left\{ \|S_m - S_{n-1}\|_p + \sum_{k=n}^{m-1} \frac{1}{k+1} \|S_k - S_{n-1}\|_p \right\} \\ &\le C_p \left(1 + \sum_{k=n}^{m-1} \frac{1}{k+1} \right) \max_{k=n,\dots,m} \|S_k - S_{n-1}\|_p. \end{aligned}$$

Thus for $n \le N \le 2n+1$

$$1 + \sum_{k=n}^{N-1} \frac{1}{k+1} \le 1 + \frac{1}{n+1}(N-n) \le 2,$$

we obtain

$$\max_{k=n,\dots,N} \|\widetilde{S}_{n-1} - \widetilde{S}_k\|_p \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p.$$

(8). From the equality

$$2\sum_{j=n}^{m} c_j e^{ijx} = \left(S_m(x) - S_{n-1}(x)\right) + i\left(\widetilde{S}_m(x) - \widetilde{S}_{n-1}(x)\right)$$

and (7) we get

$$\begin{aligned} 2 \max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_j e^{ijx} \right) \right\|_p &\le 2^{\frac{1}{p}} \left\{ \max_{m=n,\dots,N} \|S_m - S_{n-1}\|_p + \max_{m=n,\dots,N} \|\widetilde{S}_m - \widetilde{S}_{n-1}\|_p \right\} \\ &\le 2^{\frac{1}{p}} (1 + C_p) \max_{m=n,\dots,N} \|S_m - S_{n-1}\|_p, \end{aligned}$$

which is the required estimate. The estimate (9) we can prove in the same manner as the estimate (8). It is sufficient to use the equality

$$2\sum_{j=n}^{m} c_{-j} e^{-ijx} = \left(S_m(x) - S_{n-1}(x)\right) - i\left(\widetilde{S}_m(x) - \widetilde{S}_{n-1}(x)\right),$$

therefore by the reason of similarity we omit the details.

(10). By the equality

$$\begin{aligned} S_m(x) - S_{n-1}(x) &= \frac{m+1}{m}\left(S_m(x) - \sigma_m(x)\right) \\ &+ \sum_{k=n}^{m-1} \frac{1}{k}\left(S_k(x) - \sigma_k(x)\right) - \left(S_{n-1}(x) - \sigma_{n-1}(x)\right) \end{aligned}$$

we have that

$$\begin{aligned} \|S_m - S_{n-1}\|_p &\le 2^{\frac{1}{p}} \left\{ \frac{m+1}{m} \|S_m - \sigma_m\|_p + \sum_{k=n}^{m-1} \frac{1}{k} \|S_k - \sigma_k\|_p + \|S_{n-1} - \sigma_{n-1}\|_p \right\} \\ &= 2^{\frac{1}{p}} \left\{ \|S_m - \sigma_m\|_p + \sum_{k=n}^{m} \frac{1}{k} \|S_k - \sigma_k\|_p + \|S_{n-1} - \sigma_{n-1}\|_p \right\} \\ &\le 2^{\frac{1}{p}} \left(2 + \sum_{k=n}^{m} \frac{1}{k} \right) \max_{k=n-1,\dots,m} \|S_k - \sigma_k\|_p. \end{aligned}$$

Thus

$$\max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p \le 5 \cdot 2^{\frac{1}{p}} \max_{k=n-1,\dots,N} \|S_k - \sigma_k\|_p \quad \text{for} \quad n=1.$$

The case when $n \ge 2$ can be treated in a similar manner. Indeed, since for $n \le N \le 2n+1$ we have

$$2 + \sum_{k=n}^{m} \frac{1}{k} \le 2 + \frac{N-n+1}{n} \le 3 + \frac{2}{n} \le 4,$$

then the estimate (10) holds for all $n \ge 1$.

(11). From the estimate (8) we have

$$HL := \left\| \sum_{j=n}^{N} c_j e^{ijx} \right\|_p \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p. \tag{13}$$

On the other hand, by the Lemma 3 we obtain

$$HL := \left\| \sum_{j=n}^{N} c_j e^{ijx} \right\|_p \geq \frac{1}{C_p} \left(\sum_{k=n}^{N} \frac{|c_k|^p}{(k+1-n)^{2-p}} \right)^{1/p}. \tag{14}$$

Hence, from (13) and (14) we get

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p}{(k+1-n)^{2-p}} \right)^{1/p} \leq C_p \max_{k=n,\ldots,N} \|S_k - S_{n-1}\|_p. \tag{15}$$

In a similiar manner one can find the following estimate

$$\left(\sum_{k=n}^{N} \frac{|c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \leq C_p \max_{k=n,\ldots,N} \|S_k - S_{n-1}\|_p. \tag{16}$$

It is obvious that from (15) and (16) follows

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \leq C_p \max_{k=n,\ldots,N} \|S_k - S_{n-1}\|_p,$$

which proves the estimate (11).

(12). The equality $S_N(x) - S_{n-1}(x) = \sum_{j=n}^{N} c_j e^{ijx} + \sum_{j=n}^{N} c_{-j} e^{-ijx}$ and Lemma 3 give

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p}{(N+1-k)^{2-p}} \right)^{1/p} \leq C_p \|S_N - S_{n-1}\|_p,$$

and

$$\left(\sum_{k=n}^{N} \frac{|c_{-k}|^p}{(N+1-k)^{2-p}} \right)^{1/p} \leq C_p \|S_N - S_{n-1}\|_p.$$

Using the last two estimates we obtain

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(N+1-k)^{2-p}} \right)^{1/p} \leq C_p \|S_N - S_{n-1}\|_p.$$

This completes the proof of the Lemma 4. □

We shall prove now an another lemma which in this paper do not need us. The only its importance is that it extends the Lemma 2 in [1] from the case $p = 1$ to the case $0 < p < 1$. It may be useful for the other aspects.

Lemma 5. *For any trigonometric series (1) and arbitrary natural number n, the following estimate holds ($0 < p < 1$):*

$$\|\sigma_n - S_n\|_p \leq C_p \left\{ \frac{1}{n+1} \sum_{j=1}^{n-1} \|S_j - S_{[j/2]}\|_p + 2 \max_{k=[n/2],\ldots,n} \|S_k - S_{[n/2]}\|_p \right\}. \tag{17}$$

If

$$\max_{k=[n/2],\ldots,n} \|S_k - S_{[n/2]}\|_p = o(1) \ (= O(1)), \tag{18}$$

then condition (20) (see below in this paper) is satisfied.

Proof. Applying Lemma 1 to the equality

$$(n+1)\left(S_n(x) - \sigma_n(x)\right) = \sum_{j=1}^{n-1} \left(S_j(x) - S_{[j/2]}(x)\right) + n\left(S_n(x) - S_{[n/2]}(x)\right) - 2 \sum_{j=[n/2]+1}^{n-1} \left(S_j(x) - S_{[n/2]}(x)\right),$$

we obtain

$$\begin{aligned}(n+1)\|S_n - \sigma_n\|_p &\leq 4^{\frac{1}{p}} \left\{ \sum_{j=1}^{n-1} \|S_j - S_{[j/2]}\|_p + n\|S_n - S_{[n/2]}\|_p \right\} \\ &\quad + 2^{1+\frac{1}{p}} \sum_{j=[n/2]+1}^{n-1} \|S_j - S_{[n/2]}\|_p \leq 4^{\frac{1}{p}} \left\{ \sum_{j=1}^{n-1} \|S_j - S_{[j/2]}\|_p + n\|S_n - S_{[n/2]}\|_p \right\} \\ &\quad + 2^{1+\frac{1}{p}} (n - [n/2] - 1) \max_{k=[n/2],\ldots,n} \|S_k - S_{[n/2]}\|_p \\ &\leq 4^{\frac{1}{p}} \left\{ \sum_{j=1}^{n-1} \|S_j - S_{[j/2]}\|_p + (2n-1) \max_{k=[n/2],\ldots,n} \|S_k - S_{[n/2]}\|_p \right\}.\end{aligned}$$

Supposing that (18) holds, then from (17) obviously the estimate (20) holds. □

The main results of this paper are the following statements which extends Theorem 1 and Corollary 1 from the case $p = 1$ to the case $0 < p < 1$.

Theorem 2. *If $n \geq 2$ is an integer and $0 < p < 1$, then*

$$\left(\sum_{k=[\frac{n}{2}]}^{2n} \frac{\lambda_k(p)}{(|n-k|+1)^{2-p}} \right)^{1/p} \leq C_p \max_{k=[\frac{n}{2}]-1,\ldots,2n} \|\sigma_k - S_k\|_p. \tag{19}$$

In particular:

1. *If*

$$\|\sigma_n - S_n\|_p = o(1) \ (= O(1)), \tag{20}$$

then

$$\sum_{k=[\frac{n}{2}]}^{2n} \frac{\lambda_k(p)}{(|n-k|+1)^{2-p}} = o(1) \ (= O(1) \text{ respectively}). \tag{21}$$

2. *Assume that series (1) converges (possesses bounded partial sums) in the $L^p(0 < p < 1)-$metric; then condition (20) holds.*

Proof. From Lemma 4, according to the estimates (11) and (10)

$$\begin{aligned}\left(\sum_{k=n}^{2n} \frac{\lambda_k(p)}{(k+1-n)^{2-p}} \right)^{1/p} &\leq C_p \left(\sum_{k=n}^{2n} \frac{|c_k|^p + |c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \\ &\leq C_p \max_{k=n,\ldots,2n} \|S_k - S_{n-1}\|_p \leq C_p \max_{k=n-1,\ldots,2n} \|S_k - \sigma_k\|_p.\end{aligned} \tag{22}$$

On the other hand according to the estimates (12) and (10), for $2[n/2]+1 \geq n$ we have

$$\left(\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{\lambda_k(p)}{(n+1-k)^{2-p}}\right)^{1/p} \leq C_p \left(\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{|c_k|^p + |c_{-k}|^p}{(n+1-k)^{2-p}}\right)^{1/p} \leq C_p \left\|S_n - S_{\left[\frac{n}{2}\right]-1}\right\|_p \leq C_p \max_{k=\left[\frac{n}{2}\right]-1,\dots,n} \|S_k - \sigma_k\|_p. \tag{23}$$

Adding (22) and (23) we obtain (19). In addition, from (20) and (19) imply (21).

Let the series (1) converges (possesses bounded partial sums) in the $L^p(0<p<1)$-metric, then

$$\|\sigma_n - S_n\|_p \leq 2^{\frac{1}{p}} \left\{ \|f - S_n\|_p + \|\sigma_n - f\|_p \right\} = o(1)\ (= O(1)).$$

Therefore (20) implies (21). This completes the proof of the Theorem 3.1. □

The following corollary is a direct consequence of the Theorem 2.

Corollary 2. *1. Assume that series* (4) *or* (5) *satisfies condition* (20), *then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{|a_k|^p}{(|n-k|+1)^{2-p}} = o(1)\ \ (= O(1)\ \textit{respectively}). \tag{24}$$

2. Assume that series (4) *or* (5) *converges (possesses bounded partial sums) in the* $L^p(0<p<1)$*-metric, then condition* (24) *holds.*

REFERENCES

[1] Belov A.S. *On conditions of the average convergence (upper boundness) of trigonometric series.* J. Math. Sci. (New York) 2008, **155** (1), 5–17. doi:10.1007/s10958-008-9204-2 (translation of Theory of functions. In: Sovrem. Mat. Fundam. Napravl. 25. RUDN, Moskow, 2007.)

[2] Krasniqi Xh.Z. *On the convergence (upper boundness) of trigonometric series.* Math. Commun. 2009, **14** (2), 245–254.

[3] Osval'd P. *Some inequalities for trigonometric polynomials in the L_p metric,* $0<p<1$. Izv. Vyssh. Uchebn. Zaved. Mat. 1976, **7** (170), 65–75.

[4] Hardy G.H., Littlewood J.E. *Some new properties of fourier constants.* Math. Ann. 1927, **97** (170), 159–209. doi:10.1007/BF01447865

[5] Storozenko E.A., Krotov V.G., Osval'd P. *Direct and inverse theorems of Jackson type in the spaces $L^p, 0<p<1$.* Sb. Math. 1975, **27** (3), 355–376. doi:10.1070/SM1975v027n03ABEH002519 (translation of Mat. Sb. 1975, **98** (3), 395–415. (in Russian))

Received 02.11.2013

Краснікі Х.З. *Про необхідну умову для $L^p(0<p<1)-$збіжності (обмеженості зверху) тригонометричного ряду* // Карпатські матем. публ. — 2015. — Т.7, №1. — С. 83–90.

В цій статті доведено, що умова $\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{\lambda_k(p)}{(|n-k|+1)^{2-p}} = o(1)\ (= O(1))$, є необхідною умовою для $L^p(0<p<1)-$збіжності (обмеженості зверху) тригонометричного ряду. Результати статті узагальнюють деякі результати Бєлова А.С. [1].

Ключові слова і фрази: тригонометричний ряд, L^p-збіжність, нерівність Харді-Літтлвуда, нерівності Бернштейна-Зігмунда.

ISSN 2075-9827
Carpathian Math. Publ. 2015, **7** (1), 91–100
doi:10.15330/cmp.7.1.91-100

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.91–100

НЕНЯ О.І.

ПРО ПЕРМАНЕНТНІСТЬ ДИСКРЕТНОЇ СИСТЕМИ МОДЕЛІ ХИЖАК-ЖЕРТВА З НЕМОНОТОННОЮ ФУНКЦІЄЮ ВПЛИВУ ТА НЕСКІНЧЕННИМ ЗАПІЗНЕННЯМ

У роботі розглянуто систему рівнянь, яка є дискретним аналогом моделі хижак-жертва з немонотонною функцією впливу та нескінченним запізненням. Досліджується проблема побудови умов перманентної поведінки динамічної моделі. Умова перманентності забезпечує обмеженість розв'язків зверху та знизу, але при цьому вимагає щоб розв'язки залишалися постійно додатними. Для отримання достатніх умов перманентної поведінки розв'язків системи використано методи, які базуються на застосуванні теорем порівняння.

Ключові слова і фрази: модель хижак-жертва, перманентність, функціональний вплив.

Kyiv National Economic University, 54/1 Prospect Peremogy, 03680, Kyiv, Ukraine
E-mail: alexni@ukr.net

ВСТУП

Дослідження різноманітних питаннь динамічної взаємодії між елементами моделі хижак-жертва було та є одним з домінуючих, як в екології, так і в математичній біології [3]. Актуальними є проблеми локальної та глобальної стікості, періодичності, перманентної поведінки розв'язку моделі хижак-жертва [7, 8].

Існують численуі біологічні та фізіологічні свідоцтва [1, 2, 6], що у випадках, коли хижаки вимушені в пошуках здобичі ділитися жертвою або конкурувати за жертву, більш повною, в порівнянні з класичною моделлю, є модель, у якій темп приросту чисельності хижака має бути функцією не однієї змінної чисельності популяції жертви і не двох незалежних змінних чисельності жертв та хижаків, а однієї змінної — відношення чисельності популяції жертви до популяції хижака. Дану функцію звичайно називають трофічною функцією хижака або функціональним впливом.

У роботах [4], [5] розглядається модель хижак-жертва з нескінченним запізненням

$$\begin{cases} x'(t) = x(t)\left[a(t) - b(t)\int\limits_{-\infty}^{t} K(t-s)x(s)ds\right] - c(t)g\left(\frac{x(t)}{y(t)}\right)y(t), \\ y'(t) = y(t)\left[-d(t) + e(t)g\left(\frac{x(t-\tau(t))}{y(t-\tau(t))}\right)\right]. \end{cases} \tag{1}$$

У мікробіологічній динаміці та хімічній кінетиці функціональний вплив описує поглинання субстрату мікроорганізмами. В більшості випадків трофічна функція $g(u)$ монотонна. Хоча, існують експеременти які показують, що немонотонні впливи трапляються на мікробіологічному рівні: коли концентрація поживної речовини досягає високого рівня може трапитися ефект сповільнення зростання кількості мікроорганізмів. Таке

УДК 517.9
2010 *Mathematics Subject Classification:* 39A22, 39A60.

часто спостерігається коли мікроорганізми використовуються для непродуктивного розкладання або для водного очищення.

Для кожної обмеженої послідовності $a(n)$ введемо позначення

$$a^u = \sup_{n\in\mathbb{N}} a(n), \quad a^l = \inf_{n\in\mathbb{N}} a(n).$$

У даній роботі розглядається система рівнянь, яка є дискретним аналогом системи (1):

$$\begin{cases} x(n+1) = x(n)\exp\left\{a(n) - b(n)\sum\limits_{s=1}^{\infty} K(s)x(n-s) - c(n)g\left(\frac{x(n)}{y(n)}\right)\frac{y(n)}{x(n)}\right\}, \\ y(n+1) = y(n)\exp\left\{-d(n) + e(n)g\left(\frac{x(n-\tau(n))}{y(n-\tau(n))}\right)\right\}, n = 0,1,2,..., \end{cases} \tag{2}$$

де $x(n), y(n)$ — представляють щільності популяцій жертви та хижака, $n \geq 0$, $a(n), b(n)$, $c(n), e(n), d(n), \tau(n)$ — обмежені, невід'ємні послідовності такі, що

$$0 < a^l \leq a^u, \quad 0 < b^l \leq b^u, \quad 0 < c^l \leq c^u, \quad 0 < d^l \leq d^u, \quad 0 < e^l \leq e^u, \quad 0 < \tau^l \leq \tau^u.$$

Дискретна функція $K(\cdot)$ задовольняє наступні умови:

(H_1) $K(s) \in [0,\infty)$ і обмежена для $s = 1,2,3,\ldots$;

(H_2) $\sum\limits_{s=1}^{\infty} K(s) = 1.$

В даній роботі досліджується перманентна поведінка розв'язку $(x(n), y(n))$ системи рівнянь (2) з початковими умовами:

$$x(\theta) = \varphi_1(\theta), \quad y(\theta) = \varphi_2(\theta), \quad \varphi_i(0) > 0, \quad \varphi_i(\theta) \geq 0, \quad i = 1,2, \tag{3}$$

для $\theta \in \mathbb{Z}^- = \{\ldots,-2,-1,0\}$.

Для системи (2) з додатними початковими умовами (3) розв'язок $(x(n), y(n))$ існує для всіх $n \geq 0$, може бути однозначно побудований послідовно і, згідно з видом рівнянь системи (2), задовольняє умови $x(n) > 0, y(n) > 0, n \geq 0$.

Функція $g(u)$ системи (2) є немонотонною і задовольняє таким умовам (**NM**):

(i) $g \in C^1([0, +\infty), \mathbb{R}), g(0) = 0$;

(ii) існує така стала $p > 0$, що $(u-p)g'(u) < 0$ для $u \neq p$;

(iii) $\lim\limits_{u\to+\infty} g(u) = 0$;

(iv) $h'(u) < 0$ для всіх $u \geq 0$, та $h(0) = \lim\limits_{u\to 0}\frac{g(u)}{u}$, де $h(u) = \frac{g(u)}{u}$.

Розглянемо функцію

$$g(u) = \frac{\beta u^{\alpha-1}}{\gamma + u^\alpha}, \quad \alpha \geq 2,$$

яка, як неважко пересвідчитись, задовольняє умови (i)–(iv).

На основі умов (**NM**) можна довести, що при виконанні нерівності $d^u < e^l g(p)$ рівняння $g(u) = \frac{d^u}{e^l}$ має два додатні корені $0 < r_1 < r_2$.

Допоміжні результати

Означення 1. *Систему рівнянь (2) будемо називати перманентною, якщо існують додатні сталі $m_i, M_i, i = 1,2$ такі, що*

$$m_1 \leq \lim_{n\to+\infty}\inf x(n) \leq \lim_{n\to+\infty}\sup x(n) \leq M_1,$$

$$m_2 \leq \lim_{n\to+\infty}\inf y(n) \leq \lim_{n\to+\infty}\sup y(n) \leq M_2,$$

для будь якого розв'язку $u(n) = (x(n), y(n))$ системи (2) з додатними початковими даними.

Поняття перманентності грає важливу роль у математичній біології. Біологічно це означає, що коли система при взаємодії різних видів стала у певному сенсі, то всі види виживають у довгостроковому проміжку часу.

Лема 1 ([9]). *Нехай $\{x(n)\}$ задовольняє умови $x(n) > 0$ та*

$$x(n+1) \leq x(n)exp\left(r(n)\left(1 - ax(n)\right)\right)$$

для $n \in [n_1, \infty)$, де $a > 0$, $\{r(n)\}$ — додатна послідовність. Тоді

$$\lim_{n\to+\infty}\sup x(n) \leq \frac{1}{ar^u}exp\left\{r^u - 1\right\}.$$

Лема 2 ([9]). *Нехай $\{x(n)\}$ задовольняє умови:*

$$x(n+1) \geq x(n)exp\left(r(n)\left(1 - ax(n)\right)\right), \quad n \geq N_0,$$

та $\lim\limits_{n\to+\infty}\sup x(n) \leq M$, $x(N_0) > 0$, $N_0 \in \mathbb{N}$, де $aM > 1$, $\{r(n)\}$ — додатна послідовність. Тоді $\lim\limits_{n\to+\infty}\inf x(n) \geq \frac{1}{a}exp\left\{r^u\left(1 - aM\right)\right\}$.

Лема 3 ([10]). *Нехай $\{x(n)\}$, $\{b(n)\}$ — невід'ємні послідовності, визначені на $\mathbb{N}$, $c \geq 0$ — стала. Якщо*

$$x(n) \leq c + \sum_{s=0}^{n-1} b(s)x(s), \quad n \in \mathbb{N},$$

то

$$x(n) \leq c\prod_{s=0}^{n-1}\left[1 + b(s)\right], \quad n \in \mathbb{N}.$$

Лема 4. *Нехай для $\{x(n)\}$ виконуються умови $x(n) > 0$ та*

$$x(n+1) \geq x(n)exp\left\{r(n)\left(-1 + ag\left(\frac{K}{x(n)}\right)\right)\right\} \tag{4}$$

для $n \in [n_1, \infty)$, де $\{r(n)\}$ — додатна послідовність, $a > 0$, $K > 0$, $g'(u) > 0$ при $u < p$. Тоді

$$\lim_{n\to+\infty}\inf x(n) \geq \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}exp\left\{-r^u\right\} \tag{5}$$

при $g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p} < p$.

Доведення. Нехай існує таке $l_0 \in [n_1, +\infty)$, що $x(l_0+1) < x(l_0)$. Тоді з (4) випливає, що

$$x(l_0) > \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}.$$

Використовуючи останню нерівність отримуємо таку оцінку:

$$\begin{aligned} x(l_0+1) &\geq x(l_0)exp\left\{r(l_0)\left(-1+ag\left(\frac{K}{x(l_0)}\right)\right)\right\} \\ &\geq x(l_0)exp\left\{r^u\left(-1+ag\left(\frac{K}{x(l_0)}\right)\right)\right\} \\ &> x(l_0)exp\{-r^u\} \geq \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}exp\{-r^u\} = m. \end{aligned} \tag{6}$$

Доведемо, що $x(n) \geq m$ для всіх $n \in [l_0, +\infty)$. Припустимо, що існує число $\tilde{p}_0 \in [l_0, +\infty)$ таке, що $x(\tilde{p}_0) < m$. Тоді $\tilde{p}_0 \geq l_0 + 2$. Нехай p_0 — найменше ціле число таке що $x(p_0) < m$. Тоді $x(p_0 - 1) > x(p_0)$, звідки випливає, що застосувавши вищеприведені перетворення до $x(p_0)$, отримаємо, що $x(p_0) \geq m$. Отримуємо протиріччя.

Розглянемо випадок, коли $x(n+1) > x(n)$ для всіх $n \in [n_1, +\infty)$. Нехай існує $\lim\limits_{n\to+\infty} x(n) = L$. Стверджуємо, що

$$L \geq \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}.$$

Припустимо протилежне: $L < \frac{K}{g^{-1}\left(\frac{1}{a}\right)\big|_{u<p}}$. Тоді існує число $N_0 \in \mathbb{N}$ таке, що

$$x(n) < \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}$$

для всіх $n > N_0$. З цього випливає, що

$$x(n+1) \geq x(n)exp\left\{r^u\left(-1+ag\left(\frac{K}{x(n)}\right)\right)\right\}. \tag{7}$$

Перейшовши до границі в (7), одержимо:

$$\lim_{n\to+\infty} x(n) \geq \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}.$$

Отримуємо протиріччя.

Враховуючи, що $exp(-r^u) < 1$ для $r^u > 0$, маємо:

$$\lim_{n\to+\infty} x(n) \geq \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}} \geq \frac{K}{g^{-1}\left(\frac{1}{a}\right)\Big|_{u<p}}exp(-r^u), \tag{8}$$

що і доводить справедливість твердження (5). □

Основні результати

Теорема 1. *Якщо виконується умова*

$$a^l - b^u M_1 - c^u h(0) > 0, \tag{9}$$

де

$$M_1 = \frac{\exp\{a^u - 1\}}{b^l \sum\limits_{s=1}^{\infty} K(s)\exp\{-sa^u\}},$$

тоді існує таке число

$$m_1 = \frac{(a^l - c^u h(0))\exp\{a^u - c^l h(0) - \exp\{a^u - c^l h(0) - 1\}\}}{b^u \sum\limits_{s=1}^{\infty} K(s)\exp\{-s(a^l - b^u M_1 - c^u h(0))\}}, \tag{10}$$

що для розв'язку $x(n)$ системи (2) виконуються оцінки:

$$m_1 \leq \lim_{n\to+\infty} \inf x(n) \leq \lim_{n\to+\infty} \sup x(n) \leq M_1.$$

Доведення. Розглянемо випадок коли $g'(u) > 0$ для всіх $u < p$, де $u = \frac{x(n)}{y(n)}$. Тоді з першого рівняння системи (2) маємо:

$$x(n+1) \leq x(n)exp\{a^u - c^l h(p)\}.$$

Звідси отримуємо нерівність

$$x(n) \leq x(n-s)exp\{s(a^u - c^l h(p))\},$$

з якої випливає, що

$$x(n-s) \geq x(n)exp\{-s(a^u - c^l h(p))\}.$$

Підставивши останню нерівність у перше рівняння системи (2), отримуємо:

$$\begin{aligned} &x(n+1) \\ &\leq x(n)\exp\left\{a(n) - b(n)\sum_{s=1}^{\infty} K(s)x(n)\exp\left\{-s(a^u - c^l h(p))\right\} - c(n)h(p)\right\} \leq x(n) \\ &\times \exp\left\{(a(n) - c(n)h(p))\left(1 - \frac{b^l}{a^u - c^l h(p)}\sum_{s=1}^{\infty} K(s)x(n)\exp\left\{-s(a^u - c^l h(p))\right\}\right)\right\}. \end{aligned} \tag{11}$$

З умови (9), маємо $a^l - c^u h(p) \geq a^l - b^u M_1 - c^u h(0) > 0$; тому застосовуючи Лему 1 до нерівності (11) маємо наступну оцінку:

$$\lim_{n\to\infty} \sup x(n)\Big|_{u<p} \leq \frac{\exp\{a^u - c^l h(p) - 1\}}{b^l \sum\limits_{s=1}^{\infty} K(s)\exp\{-s(a^u - c^l h(p))\}} = M_1^*.$$

Розглянемо випадок, коли $g'(u) < 0$ для всіх $u > p$. З першого рівняння системи (2) маємо:

$$x(n+1) \leq x(n)exp\{a^u\}.$$

Тоді

$$x(n) \leq x(n-s)exp\{sa^u\},$$

звідки отримуємо

$$x(n-s) \geq x(n)exp\{-sa^u\}.$$

Підставивши останню нерівність у перше рівняння системи (2) отримуємо:

$$\begin{aligned} x(n+1) &\leq x(n)\exp\left\{a(n) - b(n)\sum_{s=1}^{\infty} K(s)x(n)\exp\{-sa^u\}\right\} \\ &\leq x(n)\exp\left\{a(n)\left(1 - \frac{b^l}{a^u}\sum_{s=1}^{\infty} K(s)x(n)\exp\{-sa^u\}\right)\right\}. \end{aligned} \tag{12}$$

Застосовуючи Лему 1 до нерівності (12) маємо наступну оцінку:

$$\lim_{n\to\infty}\sup x(n)\Big|_{u>p} \le \frac{\exp\{a^u-1\}}{b^l\sum\limits_{s=1}^{\infty}K(s)\exp\{-sa^u\}} = M_1^+.$$

Взявши $\max(M_1^*, M_1^+) = M_1^+ = M_1$ отримаємо оцінку:

$$\lim_{n\to+\infty}\sup x(n) \le M_1. \tag{13}$$

Нехай $g'(u) > 0$ для всіх $u < p$. З оцінки (13) випливає, що для довільного $\varepsilon > 0$ існує таке $N_1 > 0$, $N_1 \in \mathbb{N}$, що $x(n) \le M_1 + \varepsilon$ для всіх $n > N_1$. Тому з першого рівняння системи (2) маємо:

$$\begin{aligned} x(n+1) &\ge x(n)exp\{a(n) - b(n)(M_1+\varepsilon) - c(n)h(0)\} \\ &\ge x(n)exp\{a^l - b^u(M_1+\varepsilon) - c^u h(0)\}; \end{aligned}$$

звідки отримуємо

$$x(n-s) \le x(n)exp\{-s(a^l - b^u(M_1+\varepsilon) - c^u h(0))\}.$$

Підставивши останню нерівність у перше рівняння системи (2), отримуємо:

$$\begin{aligned} &x(n+1) \ge x(n) \\ &\times \exp\left\{a(n) - b(n)\sum_{s=1}^{\infty}K(s)x(n)\exp\left\{-s(a^l - b^u(M_1+\varepsilon) - c^u h(0))\right\} - c(n)h(0)\right\} \\ &\ge x(n)\exp\Bigg\{(a(n) - c(n)h(0)) \\ &\times\left(1 - \frac{b^u}{a^l - c^u h(0)}\sum_{s=1}^{\infty}K(s)x(n)\exp\left\{-s(a^l - b^u(M_1+\varepsilon) - c^u h(0))\right\}\right)\Bigg\}. \end{aligned} \tag{14}$$

З умови (9) маємо $a^l - c^u h(0) \ge a^l - b^u M_1 - c^u h(0) > 0$, тому, застосовуючи Леми 1 та 2 до нерівності (14), отримаємо наступну оцінку при $\varepsilon \to 0$:

$$\begin{aligned} \lim_{n\to\infty}\inf x(n)\Big|_{u<p} &\ge \frac{a^l - c^u h(0)}{b^u\sum\limits_{s=1}^{\infty}K(s)\exp\{-s(a^l - b^u M_1 - c^u h(0))\}} \\ &\times \exp\left\{a^u - c^l h(0) - \exp\left\{a^u - c^l h(0) - 1\right\}\right\} = m_1^*. \end{aligned}$$

Умова $aM > 1$ леми 2 набуде вигляду

$$\frac{exp\{a^u - c^l h(0) - 1\}}{a^u - c^l h(0)} > 1. \tag{15}$$

Оскільки $e^{(x-1)} \ge x$ для всіх $x \in \mathbb{R}$, то звідси випливає, що нерівність (15) виконується.

З першого рівняння системи (2) при $u > p$ маємо:

$$\begin{aligned} x(n+1) &\ge x(n)exp\{a(n) - b(n)(M_1+\varepsilon) - c(n)h(p)\} \\ &\ge x(n)exp\{a^l - b^u(M_1+\varepsilon) - c^u h(p)\}. \end{aligned}$$

Звідки випливає, що

$$x(n-s) \le x(n)exp\{-s(a^l - b^u(M_1+\varepsilon) - c^u h(p))\}.$$

Підставивши останню нерівність у перше рівняння системи (2), отримаємо:

$$\begin{aligned} &x(n+1) \ge x(n) \\ &\times \exp\left\{a(n) - b(n)\sum_{s=1}^{\infty}K(s)x(n)\exp\left\{-s(a^l - b^u(M_1+\varepsilon) - c^u h(p))\right\} - c(n)h(p)\right\} \\ &\ge x(n)\exp\Bigg\{(a(n) - c(n)h(p)) \\ &\times\left(1 - \frac{b^u}{a^l - c^u h(p)}\sum_{s=1}^{\infty}K(s)x(n)\exp\left\{-s(a^l - b^u(M_1+\varepsilon) - c^u h(p))\right\}\right)\Bigg\}. \end{aligned} \tag{16}$$

З умови (9) маємо $a^l - c^u h(p) \ge a^l - b^u M_1 - c^u h(p) \ge a^l - b^u M_1 - c^u h(0) > 0$. Тому, застосовуючи Леми 1 та 2 до нерівності (16), отримаємо наступну оцінку при $\varepsilon \to 0$

$$\begin{aligned} \lim_{n\to\infty}\inf x(n)\Big|_{u>p} &\ge \frac{a^l - c^u h(p)}{b^u\sum\limits_{s=1}^{\infty}K(s)\exp\{-s(a^l - b^u M_1 - c^u h(p))\}} \\ &\times \exp\left\{a^u - c^l h(p) - \exp\left\{a^u - c^l h(p) - 1\right\}\right\} = m_1^+. \end{aligned}$$

Взявши $\min(m_1^*, m_1^+) = m_1^* = m_1$ отримаємо число (10) та оцінку

$$\lim_{n\to+\infty}\inf x(n) \ge m_1. \tag{17}$$

□

Розглянемо друге рівняння системи (2) при $\tau(n) = k$.

Теорема 2. *Якщо виконуються умови*

$$\frac{d^u}{e^l} < g(p) \tag{18}$$

та

$$a^u - b^l m_1 + d^u > 0, \tag{19}$$

то існують такі числа

$$M_2 = \exp\left\{2(e^u g(p) - d^l)\right\}$$

та

$$m_2 = min(m_2^*, m_2^+), \tag{20}$$

де

$$m_2^* = \frac{m_1\exp\left\{k\left(e^l g\left(\frac{m_1}{M_2}\right) - d^u\right)\right\}}{g^{-1}\left(\frac{d^u}{e^l}\right)\Big|_{u<p}}\exp\{-d^u\},$$

$$m_1^+ = \frac{m_1}{\exp\{2(a^u - b^l m_1 + d^u)\}},$$

що для розв'язку $y(n)$ системи (2) виконуються оцінки

$$m_2 \le \lim_{n\to+\infty}\inf y(n) \le \lim_{n\to+\infty}\sup y(n) \le M_2.$$

Доведення. З другого рівняння системи (2) для всіх $u \geq 0$ маємо:

$$y(n+1) \leq y(n) exp\{-d^l + e^u g(p)\}. \quad (21)$$

Зробивши заміну змінних $z(n) = \ln y(n)$ в (21) отримуємо нерівність:

$$z(n+1) \leq z(n) + e^u g(p) - d^l. \quad (22)$$

Якщо взяти $c = e^u g(p) - d^l$, $b(s) = \begin{cases} 0, & 0 \leq s \leq n-1, \\ 1, & s = n, \end{cases}$ нерівність (22) набуде вигляду:

$$z(n+1) \leq \sum_{s=0}^{n} b(s) z(s) + c. \quad (23)$$

Врахувавши умову (18) $\frac{d^l}{e^u} < \frac{d^u}{e^l} < g(p)$ та застосувавши Лему 3 до нерівності (23), отримаємо оцінку розв'язку $z(n)$:

$$z(n) \leq 2(e^u g(p) - d^l),$$

а отже і оцінку розв'язку $y(n)$:

$$\lim_{n \to +\infty} \sup y(n) \leq \exp\left\{2(e^u g(p) - d^l)\right\} = M_2. \quad (24)$$

Розглянемо випадок, коли $g'(u) > 0$ для всіх $u < p$.

З оцінок (17) та (24) випливає, що для довільного $\varepsilon > 0$ існує таке $N_1 > 0$, $N_1 \in \mathbb{N}$, що для всіх $n > N_1$ виконуються оцінки $x(n) \geq m_1 - \varepsilon$ та $y(n) \leq M_2 + \varepsilon$.

Тому з другого рівняння системи (2) маємо:

$$y(n+1) \geq y(n) exp\{e^l g\left(\frac{m_1 - \varepsilon}{M_2 + \varepsilon}\right) - d^u\};$$

звідки отримуємо оцінку:

$$y(n-k) \leq y(n) exp\left\{-k\left(e^l g\left(\frac{m_1 - \varepsilon}{M_2 + \varepsilon}\right) - d^u\right)\right\}.$$

Підставивши останню нерівність у друге рівняння системи (2), отримуємо:

$$\begin{aligned} y(n+1) &\geq y(n) \exp\left\{e(n) g\left(\frac{m_1 - \varepsilon}{y(n) exp\left\{-k\left(e^l g\left(\frac{m_1-\varepsilon}{M_2+\varepsilon}\right) - d^u\right)\right\}}\right) - d(n)\right\} \\ &\geq y(n) \exp\left\{d(n)\left(-1 + \frac{e^l}{d^u} g\left(\frac{(m_1 - \varepsilon) exp\left\{k\left(e^l g\left(\frac{m_1-\varepsilon}{M_2+\varepsilon}\right) - d^u\right)\right\}}{y(n)}\right)\right)\right\}. \end{aligned} \quad (25)$$

Застосовуючи лему 4 до нерівності (25) та врахувавши умову (18) отримаємо наступну оцінку при $\varepsilon \to 0$:

$$\lim_{n \to \infty} \inf y(n)\Big|_{u<p} \geq \frac{m_1 \exp\left\{k\left(e^l g\left(\frac{m_1}{M_2}\right) - d^u\right)\right\}}{g^{-1}\left(\frac{d^u}{e^l}\right)\Big|_{u<p}} \exp\{-d^u\} = m_2^*.$$

Розглянемо випадок, коли $u > p$. Введемо заміну $z(n) = \frac{x(n)}{y(n)}$. Тоді з системи (2) маємо:

$$z(n+1) = z(n) \exp\left\{a(n) - b(n) \sum_{s=1}^{\infty} K(s) x(n-s) - c(n) \frac{g(z(n))}{z(n)} + d(n) - e(n) g(z(n-k))\right\}.$$

Звідси випливає, що

$$z(n+1) \leq z(n) exp\{a^u - b^l(m_1 - \varepsilon) + d^u\}.$$

Зробивши заміну змінних $\xi(n) = \ln z(n)$ в останній нерівності отримуємо:

$$\xi(n+1) \leq \sum_{s=0}^{n} b(s) \xi(s) + c, \quad (26)$$

де $c = a^u - b^l(m_1 - \varepsilon) + d^u$, $b(s) = \begin{cases} 0, & 0 \leq s \leq n-1, \\ 1, & s = n. \end{cases}$

Врахувавши умову (19) та застосувавши Лему 3 до нерівності (26), отримаємо оцінку розв'язку $\xi(n)$ при $\varepsilon \to 0$:

$$\xi(n) \leq 2(a^u - b^l m_1 + d^u),$$

а отже і оцінку розв'язку $z(n)$:

$$\lim_{n \to +\infty} \sup z(n) \leq \exp\left\{2(a^u - b^l m_1 + d^u)\right\} = M_2^*. \quad (27)$$

З оцінки (27) випливає, що для довільного $\varepsilon > 0$ існує таке $N_1 > 0$, $N_1 \in \mathbb{N}$, що для всіх $n > N_1$ виконується $\frac{x(n)}{y(n)} \leq M_2^* + \varepsilon$. Звідси випливає:

$$y(n) \geq \frac{x(n)}{M_2^* + \varepsilon} \geq \frac{m_1 - \varepsilon}{M_2^* + \varepsilon}.$$

При $\varepsilon \to 0$ маємо:

$$\lim_{n \to +\infty} \inf y(n) \geq \frac{m_1}{\exp\{2(a^u - b^l m_1 + d^u)\}} = m_2^+.$$

Взявши $\min(m_2^*, m_2^+)$, отримаємо число (20) та оцінку:

$$\lim_{n \to +\infty} \inf y(n) \geq m_2.$$

□

Висновки

У роботі досліджено властивість перманентності системи різницевих рівнянь моделі хижак-жертва з немонотонною функцією впливу та нескінченним запізненням. На основі теорем порівняння побудовано нові умови перманентної поведінки динамічної моделі.

Відкритими залишаються питання побудови оцінок розв'язку $y(n)$ рівняння хижака системи (2) при $\tau(n) \neq const$ та покращення отриманих у роботі умов та оцінок.

References

[1] Arditi R., Ginzburg L.R. *Coupling in predator-prey dynamics: Ratio-dependence.* J. Theoret. Biol. 1989, **139** (3), 311–326. doi:10.1016/S0022-5193(89)80211-5

[2] Arditi R., Saiah H. *Empirical evidence of the role of heterogeneity in ratio-dependent consumption.* Ecology 1992, **73** (5), 1544–1551. doi:10.2307/1940007

[3] Berryman A.A. *The origins and evolution of predator–prey theory.* Ecology 1992, **75** (5), 1530–1535. doi:10.2307/1940005

[4] Fan Y.H., Li W.T., Wang L.L. *Periodic solutions of delayed ratio-dependent predator-prey models with monotonic or nonmonotonic functional responses.* Nonlinear Anal. Real World Appl. 2004, **5** (2), 247–263. doi:10.1016/S1468-1218(03)00036-1.

[5] Fan M., Wang K. *Periodicity in a delayed ratio-dependent predator–prey system.* J. Math. Anal. Appl. 2001, **262** (1), 179–190. doi:10.1006/jmaa.2001.7555.

[6] Gutierrez A.P. *The physiological basis of ratio-dependent predator-prey theory: A metabolic pool model of Nicholson's blowflies as an example.* Ecology 1992, **73** (5), 1552–1563. doi:10.2307/1940008

[7] Hsu S.B., Hwang T.W., Kuang Y. *Global analysis of Michaelis-Menten type ratio-dependent predator-prey system.* J. Math. Biol. 2003, **42** (6), 489–506. doi:10.1007/s002850100079.

[8] Kuang Y., Beretta E. *Global qualitative analysis of a ratio-dependent predator-prey systems.* J. Math. Biol. 1998, **36** (4), 389–406. doi:10.1007/s002850050105

[9] Yang X.T. *Uniform persistence and periodic solutions for a discrete predator–prey system with delays.* J. Math. Anal. 2006, **316** (1), 161–177. doi:10.1016/j.jmaa.2005.04.036.

[10] Yang W.S., Li X.P. *Permanence for a delayed discrete ratio-dependent predator–prey model with monotonic functional responses.* Nonlinear Anal. 2009, **10** (2), 1068—1072. doi:10.1016/j.nonrwa.2007.11.022.

Надійшло 05.04.2014

Nenya O. I. *Permanence of a discrete predator-prey system with nonmonotonic functional responses and endless delay.* Carpathian Math. Publ. 2015, **7** (1), 91–100.

A discrete-time analogue of predator-prey model with nonmonotonic functional responses and endless delay is considered in the paper. We investigate the question of obtaining conditions of permanent behavior of the dynamic model. The condition of permanence provides the limiting of the solutions but it requires the positiveness of the solutions. Sufficient conditions of permanence are obtained when the functional response function is nonmonotonic. The methods based on the estimation theorems are used to receive the sufficient permanent conditions of the solutions. These results are applied to some special population model with endless delay, some new results are obtained.

Key words and phrases: predator-prey model, permanence, functional response function.

ISSN 2075-9827

Carpathian Math. Publ. 2015, 7 (1), 101–107

doi:10.15330/cmp.7.1.101-107

http://www.journals.pu.if.ua/index.php/cmp

Карпатські матем. публ. 2015, Т.7, №1, С.101–107

OSYPCHUK M.M.

ON SOME PERTURBATIONS OF A STABLE PROCESS AND SOLUTIONS OF THE CAUCHY PROBLEM FOR A CLASS OF PSEUDO- DIFFERENTIAL EQUATIONS

A fundamental solution of some class of pseudo-differential equations is constructed by a method based on the theory of perturbations. We consider a symmetric α-stable process in multidimensional Euclidean space. Its generator **A** is a pseudo-differential operator whose symbol is given by $-c|\lambda|^{\alpha}$, where the constants $\alpha \in (1,2)$ and $c > 0$ are fixed. The vector-valued operator **B** has the symbol $2ic|\lambda|^{\alpha-2}\lambda$. We construct a fundamental solution of the equation $u_t = (\mathbf{A} + (a(\cdot), \mathbf{B}))u$ with a continuous bounded vector-valued function a.

Key words and phrases: stable process, Cauchy problem, pseudo-differential equation, transition probability density.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine

E-mail: mykhailo.osypchuk@pu.if.ua

INTRODUCTION

Let **A** denote a pseudo-differential operator that acts on a twice continuously differentiable bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$ according to the following rule

$$(\mathbf{A}\varphi)(x) = \frac{c}{\varkappa}\int_{\mathbb{R}^d} \frac{\varphi(x+y) - \varphi(x) - (y, \nabla\varphi(x))}{|y|^{d+\alpha}}\, dy, \tag{1}$$

where $c > 0$, $1 < \alpha < 2$, $d \in \mathbb{N}$ are some constants, $\varkappa = -\dfrac{2\pi^{\frac{d-1}{2}}\Gamma(2-\alpha)\Gamma\left(\frac{\alpha+1}{2}\right)\cos\frac{\pi\alpha}{2}}{\alpha(\alpha-1)\Gamma\left(\frac{d+\alpha}{2}\right)}$ and ∇ is the Hamilton operator (gradient). Here $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^d$.

It is known that the function $u(t,x) = \int_{\mathbb{R}^d} \varphi(y) g(t,x,y)\, dy$, where

$$g(t,x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(y-x,\lambda) - ct|\lambda|^{\alpha}}\, d\lambda, \tag{2}$$

is a solution of the following Cauchy problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \mathbf{A}_x u(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0+,x) &= \varphi(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{3}$$

for any bounded continuous function $(\varphi(x))_{x\in\mathbb{R}^d}$.

УДК 517.9, 519.2

2010 *Mathematics Subject Classification:* 35S11, 60G52.

If an operator acts on a function of several arguments, then it will be provided by a corresponding subscript, for example, $\mathbf{A}_x$ in (3) means that the operator $\mathbf{A}$ is acting on $u(t,x)$ as the function of the variable x.

Note, that the function $(g(t,x,y))_{t>0,x\in\mathbb{R}^d,y\in\mathbb{R}^d}$ serves as transition probability density of a Markov process in $\mathbb{R}^d$, called a symmetric stable process. The operator $\mathbf{A}$ is the generator of it.

Let us consider the equation

$$\frac{\partial u(t,x)}{\partial t} = \mathbf{A}_x u(t,x) + (a(x), \mathbf{B}_x u(t,x)), \quad t>0,\ x\in\mathbb{R}^d, \tag{4}$$

with some $\mathbb{R}^d$-valued function $(a(x))_{x\in\mathbb{R}^d}$ and d-dimensional pseudo-differential operator $\mathbf{B}$ of the order less than α.

In this article, we consider the case, where the a is a bounded continuous function and the operator $\mathbf{B}$ is defined on a differentiable bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$ by the equality

$$(\mathbf{B}\varphi)(x) = \frac{2c}{\alpha\varkappa}\int_{\mathbb{R}^d}\frac{\varphi(x+y)-\varphi(x)}{|y|^{d+\alpha}}\,y\,dy.$$

Note, that $\mathbf{A} = \frac{1}{2}\mathbf{div}(\mathbf{B})$.

We construct a fundamental solution of equation (4) by perturbing the transition probability density of a symmetric stable process. The fundamental solution of equation (4) was constructing in [2] under the assumption that the function a satisfied Holder's condition.

Symmetric stable processes were perturbed by terms of the type $(a(x),\nabla)$ under various assumptions on the function $(a(x))_{x\in\mathbb{R}^d}$ in many papers (see, for example, [1, 3, 5, 6]). The perturbation of stable processes with delta-function in coefficient is constructed in [4].

1 Perturbation of a stable process

We consider a function $(G(t,x,y))_{t>0,x\in\mathbb{R}^d,y\in\mathbb{R}^d}$ as a result of perturbing the transition probability density $g(t,x,y)$ of a symmetric stable process, if it is a solution of the following equation

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,z)(\mathbf{B}_z G(\tau,z,y), a(z))\,dz. \tag{5}$$

Now we define a function $(e(x))_{x\in\mathbb{R}^d}$ by the equality $e(x) = \frac{1}{|a(x)|}a(x)$ for $x\in\mathbb{R}^d$ such that $|a(x)|\neq 0$ and an arbitrary value (with preservation of the measurability) otherwise. Then the equation (5) takes the form

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,z)(\mathbf{B}_z G(\tau,z,y), e(z))|a(z)|\,dz. \tag{6}$$

It is easy to establish the following equality using the representation (2) and integration by parts $\mathbf{B}_x g(t,x,y) = \frac{2}{\alpha}\frac{y-x}{t}g(t,x,y)$. Denote by $V_0(t,x,y)$ a function that is given by the equality

$$V_0(t,x,y) = (\mathbf{B}_x g(t,x,y), e(x)) = \frac{2}{\alpha}\frac{(y-x,e(x))}{t}g(t,x,y). \tag{7}$$

We will construct the solution of (6) in the form

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz, \tag{8}$$

where the function $V(t,x,y)$ satisfies the equation

$$V(t,x,y) = V_0(t,x,y) + \int_0^t d\tau\int_{\mathbb{R}^d} V_0(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz. \tag{9}$$

The equation (9) can be solved by the method of successive approximations, namely its solution will be found in the form

$$V(t,x,y) = \sum_{k=0}^{\infty} V_k(t,x,y), \tag{10}$$

where $V_0(t,x,y)$ is defined by the equality (7) and for $k\geq 1$ the following equality

$$V_k(t,x,y) = \int_0^t d\tau\int_{\mathbb{R}^d} V_0(t-\tau,x,z)V_{k-1}(\tau,z,y)|a(z)|\,dz$$

is valid.

The well-known estimate (see [2]) ($t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, and $N>0$ is a constant)

$$g(t,x,y) \leq N\frac{t}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \tag{11}$$

allows us to write down

$$|V_0(t,x,y)| \leq \frac{2}{\alpha}N\frac{|y-x|}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \leq \frac{2}{\alpha}\frac{N}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}.$$

Then, we get that the inequality

$$|V_k(t,x,y)| \leq \|a\|\frac{2N}{\alpha}\int_0^t d\tau\int_{\mathbb{R}^d}\frac{1}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha-1}}|V_{k-1}(\tau,z,y)|\,dz$$

is true, where $\|a\| = \sup_{x\in\mathbb{R}^d}|a(x)|$.

In order to estimate V_k we make use of the following inequality (see [2])

$$\int_0^t d\tau\int_{\mathbb{R}^d}\frac{1}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha-1}}\cdot\frac{\tau^\delta}{(\tau^{1/\alpha}+|z-x|)^{d+\alpha-1}}\,dz \leq C\frac{\alpha}{1+\alpha\delta}\left(1+\delta B\left(\frac{1}{\alpha},\delta\right)\right)\frac{t^{\delta+1/\alpha}}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}},$$

valid for $\delta > -1/\alpha$, where $C>0$, and $B(\cdot,\cdot)$ is the Euler beta function. We obtain for $k\geq 1$

$$|V_k(t,x,y)| \leq \frac{(2N)^{k+1}(C\|a\|)^k}{\alpha}\frac{1}{k!}\frac{t^{k/\alpha}}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}\prod_{n=1}^{k-1}\left(1+\frac{n}{\alpha}B\left(\frac{1}{\alpha},\frac{n}{\alpha}\right)\right).$$

Note, that $r_k = \frac{(2NC\|a\|t^{1/\alpha})^k}{k!}\prod_{n=1}^{k-1}\left(1+\frac{n}{\alpha}B\left(\frac{1}{\alpha},\frac{n}{\alpha}\right)\right)$ is positive and the relation

$$\lim_{k\to\infty}\frac{r_{k+1}}{r_k} = \lim_{k\to\infty}\frac{2NC\|a\|t^{1/\alpha}}{k+1}\left(1+\frac{k}{\alpha}B\left(\frac{1}{\alpha},\frac{k}{\alpha}\right)\right) = 0$$

is true. Therefore, the series on the right hand side of (10) converges uniformly in $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$ and locally uniformly in $t>0$. Thus, the function V, given by the equality (10), is a solution of the equation (9). In addition, the following inequality

$$|V(t,x,y)| \leq C_T\frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}} \tag{12}$$

is proved for $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$ and $0<t\leq T$, where C_T is a positive constant that may be depended on $T>0$.

Remark. *The constructed function $V(t,x,y)$ is the unique solution of equation* (9) *in the class of functions that satisfy inequality* (12).

Define the function $G(t,x,y)$ by the equality (8) where the function $V(t,x,y)$ is defined in (10). Then we can perform the following calculations

$$\begin{aligned}(\mathbf{B}_x G(t,x,y), e(x)) &= V_0(t,x,y) + \int_0^t d\tau \int_{\mathbb{R}^d} V_0(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz \\ &= V(t,x,y).\end{aligned}$$

We here took the possibility of applying of the operator $\mathbf{B}$ under integral, which is proved in the following Lemma.

Lemma. *The equality*

$$\mathbf{B}_x \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz = \int_0^t d\tau \int_{\mathbb{R}^d} \mathbf{B}_x g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz$$

is true.

Proof. Let us consider a set of operators $\{\mathbf{B}^\varepsilon : \varepsilon > 0\}$ that act on a continuously differentiable bounded function $(\varphi(x))_{x\in\mathbb{R}^d}$ according to the following rule

$$(\mathbf{B}^\varepsilon\varphi)(x) = \frac{2c}{\alpha\varkappa}\int_{|u|\ge\varepsilon} \frac{\varphi(x+u)-\varphi(x)}{|u|^{d+\alpha}}\, y\, dy.$$

It is clear that $\lim_{\varepsilon\to 0+}(\mathbf{B}^\varepsilon\varphi)(x) = (\mathbf{B}\varphi)(x)$ for all functions φ, described above, and $x \in \mathbb{R}^d$.

The inequalities (11) and (12) allow us to assert that

$$\begin{aligned}&\left|\frac{u}{|u|^{d+\alpha}}(g(t-\tau,x+u,z)-g(t-\tau,x,z))V(\tau,z,y)|a(z)|\right| \\ &\le \frac{const}{|u|^{d+\alpha-1}}\left(\frac{t-\tau}{((t-\tau)^{1/\alpha}+|z-x-u|)^{d+\alpha}} + \frac{t-\tau}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha}}\right) \\ &\times \frac{1}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha-1}}.\end{aligned}$$

It is easy to see that the right hand side of this inequality is the integrable function with respect to (u,τ,z) on the set $\{|u|\ge\varepsilon\}\times(0;t)\times\mathbb{R}^d$ for all $t>0$ and $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$. Here we used the results of [2, Lemma 5], where it is proved that

$$\begin{aligned}&\int_0^t d\tau \int_{\mathbb{R}^d} \frac{(t-\tau)^{\beta/\alpha}}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha+k}}\frac{\tau^{\gamma/\alpha}}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha+l}}\,dz \\ &\le C\left[B\left(\frac{\beta-k}{\alpha},1+\frac{\gamma}{\alpha}\right)t^{\frac{\beta+\gamma-k}{\alpha}}\frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha+l}}\right. \\ &\left.+ B\left(1+\frac{\beta}{\alpha},\frac{\gamma-l}{\alpha}\right)t^{\frac{\beta+\gamma-l}{\alpha}}\frac{1}{(t^{1/\alpha}+|y-x|)^{d+\alpha+k}}\right]\end{aligned} \tag{13}$$

for $-\alpha<k<\beta$, $-\alpha<l<\gamma$ and $C>0$, which depends only on d, α, k and l.

Therefore, we obtain the following equality

$$\mathbf{B}^\varepsilon_x \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz = \int_0^t d\tau \int_{\mathbb{R}^d} \mathbf{B}^\varepsilon_x g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz, \tag{14}$$

using the Fubini theorem.

The inequalities (12), (13) and $|\mathbf{B}_x g(t,x,y)| \le \dfrac{const}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}$ allow us to assert that the integral $\int_0^t d\tau \int_{\mathbb{R}^d} \mathbf{B}_x g(t-\tau,x,z)V(\tau,z,y)|a(z)|\,dz$ exists. Now we have to pass to the limit with $\varepsilon\to 0+$ in the equality (14) to complete the proof of Lemma. □

We have thus got that the function $G(t,x,y)$ is the perturbation of the transition probability density $g(t,x,y)$ of a symmetric stable process.

Considering estimates (12), (11) and inequality (13), we can write for $t\in(0;T]$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$

$$\begin{aligned}|G(t,x,y)| &\le N\frac{t}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \\ &+ NC_T\|a\|\int_0^t d\tau\int_{\mathbb{R}^d}\frac{t-\tau}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha}}\frac{1}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha-1}}\,dz \\ &\le \frac{Kt}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}}\left(1+\frac{1+t^{1/\alpha}}{t^{1/\alpha}+|y-x|}\right),\end{aligned}$$

where K is a positive constant, which depends on T, α, c, $\|a\|$ and d. Note that the right hand side of the last inequality can be estimated from above by the following expression

$$\frac{\hat{K}t^{1-1/\alpha}}{(t^{1/\alpha}+|y-x|)^{d+\alpha-1}} \le \hat{K}t^{-d/\alpha},$$

where $\hat{K} = (2T^{1/\alpha}+1)K$.

2 The fundamental solution of the Cauchy problem

It is known (see [2]) that the function $g(t,x,y)$ is the fundamental solution of the Cauchy problem (3) and, in addition, the function

$$u(t,x) = \int_{\mathbb{R}^d}\varphi(y)g(t,x,y)\,dy + \int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,y)f(\tau,y)\,dy$$

is the solution of the Cauchy problem

$$\begin{aligned}&\frac{\partial u(t,x)}{\partial t} = \mathbf{A}_x u(t,x) + f(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ &u(0+,x) = \varphi(x), \quad x\in\mathbb{R}^d,\end{aligned} \tag{15}$$

for any bounded continuous functions $(\varphi(x))_{x\in\mathbb{R}^d}$ and $(f(t,x))_{t>0,x\in\mathbb{R}^d}$. Moreover, this solution is unique in the class of functions that vanish as $|x|\to\infty$.

Thus, the function

$$\begin{aligned}U(t,x) &= \int_{\mathbb{R}^d}\varphi(y)G(t,x,y)\,dy \\ &= \int_{\mathbb{R}^d}\varphi(y)g(t,x,y)\,dy + \int_0^t d\tau\int_{\mathbb{R}^d} g(t-\tau,x,y)\int_{\mathbb{R}^d} V(\tau,y,z)\varphi(z)\,dz|a(y)|\,dy\end{aligned}$$

is the unique (in the class of functions that tends to zero at infinity) solution of the Cauchy problem (15) with $f(t,x) = \int_{\mathbb{R}^d} V(t,x,z)\varphi(z)\,dz|a(x)|$.

Now we note that $V(t,x,y) = (\mathbf{B}_x G(t,x,y), e(x))$. Then

$$f(t,x) = \int_{\mathbb{R}^d} (\mathbf{B}_x G(t,x,z), a(x))\varphi(z)\,dz = (a(x), \mathbf{B}_x U(t,x)),$$

and the function $U(t,x)$ is a solution of the Cauchy problem for the equation (4) with bounded continuous function $a(x)$ and operators $\mathbf{A}$ and $\mathbf{B}$ defined by equalities (1) and (5) respectively.

Let us prove that the function $G(t,x,y)$ satisfies the equation of Kolmogorov-Chapman

$$G(t+s,x,y) = \int_{\mathbb{R}^d} G(s,x,z)G(t,z,y)\,dz \tag{16}$$

for all $s>0$, $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$. Note, the function $g(t,x,y)$ satisfies the equation (16).

Let $(\varphi(x))_{x\in\mathbb{R}^d}$ be a continuous bounded function. Put $U(s,x,\varphi) = \int_{\mathbb{R}^d} G(s,x,y)\varphi(y)\,dy$, $u(s,x,\varphi) = \int_{\mathbb{R}^d} g(s,x,y)\varphi(y)\,dy$ and $W(s,x,\varphi) = \int_{\mathbb{R}^d} V(s,x,y)\varphi(y)\,dy$.

Note, that the function $W(t,x,\varphi)$ is the unique solution of the following equation

$$W(t,x,\varphi) = W_0(t,x,\varphi) + \int_0^t d\tau \int_{\mathbb{R}^d} V_0(t-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz, \tag{17}$$

where $W_0(s,x,\varphi) = \int_{\mathbb{R}^d} V_0(s,x,y)\varphi(y)\,dy$.

Then the function $U(s,x,\varphi)$ can be given by the equality (see (5))

$$U(t,x,\varphi) = u(t,x,\varphi) + \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz.$$

Now, let us find the function $U(t+s,x,\varphi)$. We have

$$\begin{aligned} U(t+s,x,\varphi) =& u(t+s,x,\varphi) + \int_0^{t+s} d\tau \int_{\mathbb{R}^d} g(t+s-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz \\ =& \int_{\mathbb{R}^d} g(s,x,y)u(t,y,\varphi)\,dy \\ &+ \int_{\mathbb{R}^d} g(s,x,y)\,dy \int_0^t d\tau \int_{\mathbb{R}^d} g(t-\tau,y,z)W(\tau,z,\varphi)|a(z)|\,dz \\ &+ \int_t^{s+t} d\tau \int_{\mathbb{R}^d} g(t+s-\tau,x,z)W(\tau,z,\varphi)|a(z)|\,dz \\ =& \int_{\mathbb{R}^d} g(s,x,y)U(t,y,\varphi)\,dy \\ &+ \int_0^s d\tau \int_{\mathbb{R}^d} g(s-\tau,x,z)W(t+\tau,z,\varphi)|a(z)|\,dz. \end{aligned}$$

Therefore, the function $W_t(s,x,\varphi) = W(t+s,x,\varphi)$ satisfies the equation (17), where the function φ is replaced by $U(t,\cdot,\varphi)$. Then $W(t+s,x,\varphi) = W(s,x,U(t,\cdot,\varphi))$ and we arrive at the equality $U(t+s,x,\varphi) = U(s,x,U(t,\cdot,\varphi))$ or, what is the same,

$$\begin{aligned} \int_{\mathbb{R}^d} G(t+s,x,y)\varphi(y)\,dy &= \int_{\mathbb{R}^d} G(s,x,z) \int_{\mathbb{R}^d} G(t,z,y)\varphi(y)\,dy\,dz \\ &= \int_{\mathbb{R}^d} \varphi(y)\,dy \int_{\mathbb{R}^d} G(s,x,z)G(t,z,y)\,dz. \end{aligned}$$

Then the relation (16) is proved because the function φ is an arbitrary bounded continuous one.

Next, we get $\int_{\mathbb{R}^d} G(t,x,y)\,dy = 1$ from (8) and (9), because there are obvious equalities

$$\int_{\mathbb{R}^d} g(t,x,y)\,dy = 1 \quad\text{and}\quad \int_{\mathbb{R}^d} V_0(t,x,y)\,dy = \left(\mathbf{B}_x \int_{\mathbb{R}^d} g(t,x,y)\,dy, e(x)\right) = 0$$

for all $t>0$, $x\in\mathbb{R}^d$, and the uniqueness of the solution of equation (9) leads us to the identity $\int_{\mathbb{R}^d} V(t,x,y)\,dy \equiv 0$.

Unfortunately, we can not guarantee non-negativity of the function $G(t,x,y)$ and the existence of a Markov process with the generating operator $\mathbf{A} + (a(\cdot),\mathbf{B})$.

Acknowledgement. The author is grateful to Professor M. Portenko for comments and discussions which led to the improvement of the quality of this article.

REFERENCES

[1] Bogdan K., Jakubowski T. *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators.* Comm. Math. Phys. 2007, **271** (1), 179–198. doi:10.1007/s00220-006-0178-y

[2] Kochubei A.N. *Parabolic pseudodifferential equations, hypersingular integrals, and Markov processes.* Math. USSR-Izv. 1989, **33** (2), 233–259. doi:10.1070/IM1989v033n02ABEH000825 (tranaslation of Izv. Akad. Nauk SSSR Ser. Mat. 1988, **52** (5), 909–934. (in Russian))

[3] Kurenok V.P. *A note on L_2-estimates for stable integrals with drift.* Trans. Amer. Math. Soc. 2008, **300** (2), 925–938.

[4] Loebus J.V., Portenko M.I. *On one class of perturbations of the generators of a stable process.* Theory Probab. Math. Statist. 1995, **52**, 102–111.

[5] Podolynny S.I., Portenko N.I. *On multidimensional stable processes with locally unbounded drift.* Random Oper. Stochastic Equations 1995, **3** (2), 113-124.

[6] Portenko N.I. *Some perturbations of drift-type for symmetric stable processes.* Random Oper. Stochastic Equations 1994, **2** (3), 211–224.

Received 04.02.2015

Revised 08.04.2015

Осипчук М.М. *Про деяке збурення стійкого процесу та розв'язки задачі Коші для одного класу псевдо-диференціальних рівнянь.* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 101–107.

З допомогою методу теорії збурень знайдено фундаментальний розв'язок деякого класу псевдо-диференціальних рівнянь. Розглянуто симетричний α-стійкий процес в багатовимірному евклідовому просторі. Його генератор $\mathbf{A}$ є псевдо-диференціальним оператором чий символ задається функцією $-c|\lambda|^\alpha$, де $\alpha \in (1,2)$ і $c>0$ задані сталі. Векторнозначний оператор $\mathbf{B}$ має символ $2ic|\lambda|^{\alpha-2}\lambda$. Побудовано фундаментальний розв'язок рівняння $u_t = (\mathbf{A} + (a(\cdot),\mathbf{B}))u$ з неперервною обмеженою векторнозначною функцією a.

Ключові слова і фрази: стійкий процес, задача Коші, псевдо-диференціальне рівняння, щільність ймовірності переходу.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 108–113
doi:10.15330/cmp.7.1.108-113

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.108–113

PRYIMAK H.M.

HOMOMORPHISMS AND FUNCTIONAL CALCULUS IN ALGEBRAS OF ENTIRE FUNCTIONS ON BANACH SPACES

In the paper the homomorphisms of algebras of entire functions on Banach spaces to a commutative Banach algebra are studied. In particular, it is proposed a method of constructing of homomorphisms vanishing on homogeneous polynomials of degree less or equal than a fixed number n.

Key words and phrases: Aron-Berner extension, functional calculus, algebras of analytic functions on Banach spaces.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: galja_petriv@mail.ru

1 Introduction and preliminaries

In 1951 R. Arens [1] found a way of extending the product of Banach algebra A to its bidual A'' in such a way that this bidual became itself a Banach algebra. There are two canonical ways to extend the product from A to A'' which called the Arens products. We recall definitions [2].

Let A be a commutative Banach algebra, X be a Banach space over the field of complex numbers $\mathbb{C}$.

If $x \in X$ and $\lambda \in X'$ then we write $\langle \lambda, x\rangle = \lambda(x)$. For every $a, b \in A, \lambda \in A'$ and $\Phi \in A''$ define $a.\lambda \in A', \lambda.a \in A', \lambda.\Phi \in A'$ and $\Phi.\lambda \in A'$ by:

$$a.\lambda : b \mapsto \langle \lambda, ba\rangle, \lambda.a : b \mapsto \langle \lambda, ab\rangle,$$
$$\lambda.\Phi : b \mapsto \langle \Phi, b.\lambda\rangle, \Phi.\lambda : b \mapsto \langle \Phi, \lambda.b\rangle;$$

and then define two products $\Box$ and $\Diamond$ on A'' by:

$$\langle \Phi \Box \Psi, \lambda\rangle = \langle \Phi, \Psi.\lambda\rangle, \langle \Phi \Diamond \Psi, \lambda\rangle = \langle \Psi, \lambda.\Phi\rangle (\Phi, \Psi \in A'').$$

Then $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras. We say that A is *Arens regular* if for all $\Phi, \Psi \in A''$ we have $\Phi \Box \Psi = \Phi \Diamond \Psi$.

For a given complex Banach space X, $\mathcal{P}(^nX)$ denotes the Banach space of all continuous n-homogeneous complex-valued polynomials on X. The problem of extending every element of $\mathcal{P}(^nX)$ to a continuous n-homogeneous polynomial $\widetilde{P}$ on the bidual X'' of X was first studied by Aron and Berner in 1978, who showed that such extensions always exist.

Let $B : X \times \ldots \times X \to \mathbb{C}$ be the symmetric n-linear mapping associated to P. B can be extended to an n-linear mapping $\widetilde{B} : X'' \times \ldots \times X'' \to \mathbb{C}$. Let $(z_1, \ldots, z_n) \in X'' \times \ldots \times X''$.

УДК 517.98
2010 *Mathematics Subject Classification:* 46G20, 46E25.
Supported by the Grand of the President of Ukraine GP/F56/176, 2014

For a net (x_{α_k}) from X which converges to z_k in the weak-star topology of X'' for each fixed $k, 1 \leqslant k \leqslant n$, we put

$$\widetilde{B}(z_1, \ldots, z_n) = \lim_{\alpha_1} \ldots \lim_{\alpha_n} B(x_{\alpha_1}, \ldots, x_{\alpha_n}).$$

Then *the Aron-Berner extension* P on X'' to X is defined as

$$\widetilde{P}(z) = \widetilde{B}(z, \ldots, z),$$

where B is a unique continuous n-linear symmetric form for which $P(x) = B(x, \ldots, x)$ for each $x \in X$.

Consider the complete projective tensor product $A \otimes_\pi X$. Every element of $A \otimes_\pi X$ can be represented by the form $\overline{a} = \sum_k a_k \otimes_\pi x_k$, where $a_k \in A$, $x_k \in X$. For every $\overline{a} \in A \otimes_\pi X$ and $f \in H_b(X)$ (algebra of entire analytic functions of bounded type on a Banach space X) let us define $\overline{f}(\overline{a})$ in the means of functional calculus for analytic functions on a Banach spaces ([5]). Then $\widetilde{\overline{f}}$ is the Aron-Berner extension of $\overline{f}$.

In [6] using the Aron-Berner extension and approach developed in [4] it was obtained a method to construct nontrivial complex homomorphisms of $H_b(X)$ vanishing on homogeneous polynomials of degree less or equal that a fixed number n. In this paper we extend this result for Banach algebra valued homomorphism.

2 Main results

Recall that X is *a left A-module* (X is *a left module over A*), if exists a bilinear map $A \times X \to X$, $(a, x) \mapsto a \cdot x$ such that $(a_1 \cdot a_2) \cdot x = a_1 \cdot (a_2 \cdot x)$, where $a_1, a_2 \in A, x \in X$. It is easy to prove that $A \otimes_\pi X$ is a left A-module. So, using Theorem 2 ([3], p.297) we can easy obtain the following proposition.

Proposition 1. *$(A \otimes_\pi X)''$ is a left A''-module.*

In [7] it is proved a theorem about a homomorphism of algebras $H_b(X)$ and $H_b((A \otimes_\pi X''), A)$ in the case when A is some finite dimensional algebra with identity. The following theorem extends this result for the case of an infinite dimensional algebra A.

Proposition 2. *Let A be the Arens regular Banach algebra. For every $f \in H_b(X)$ there exists a function $\widetilde{\overline{f}} \in H_b((A \otimes_\pi X)'', A'')$ such that $\widetilde{\overline{f}}(e \otimes x) = ef(x), x \in X$ and the mapping $F : f \mapsto \widetilde{\overline{f}}$ is a homomorphism between algebras $H_b(X)$ and $H_b((A \otimes_\pi X)'', A'')$.*

The proof it easy follows from the fact that both the Aron-Berner extension and functional calculus are topological homomorphisms ([4], [5]).

Example 1. *Let us show that in the case if A is not Arens regular, then the map F is not necessary a homomorphism. Let $A = \ell_1, X = \mathbb{C}^2$. We need to prof that*

$$F : H_b(\mathbb{C}^2) \to H_b((\ell_1 \otimes_\pi \mathbb{C}^2)'', \ell_1'') \text{ the are } f, g \in H_b(\mathbb{C}^2) \text{ such that } F(fg) \neq F(f)F(g).$$

For each $t = (t_1, t_2) \in \mathbb{C}^2$ put $f(t) = t_1, g(t) = t_2$ and apply the extension operator $\mathbb{C}^2 \ni t \rightsquigarrow x \in \ell_1 \times \ell_1$ and the Aron-Berner extension $\ell_1 \times \ell_1 \ni x \rightsquigarrow u = (u_1, u_2) \in \ell_\infty \times \ell_\infty$. Then

$$\overline{f}(x) = x_1 \in \ell_1, \quad \overline{g}(x) = x_2 \in \ell_1, \quad \overline{f}(x)\overline{g}(x) = x_1 * x_2,$$

where " $*$ " is the convolution product in ℓ_1. Suppose that

$$\tilde{\bar{f}}(u) = u_1 \in \ell_1'', \qquad \tilde{\bar{g}}(u) = u_2 \in \ell_1''.$$

Then we have $\tilde{\bar{f}}(u)\tilde{\bar{g}}(u) = u_1 \Box u_2$ and $\tilde{\bar{g}}(u)\tilde{\bar{f}}(u) = u_1 \Diamond u_2 = u_1 \Box u_2$.

Since $u_1 \Diamond u_2 \neq u_1 \Box u_2$ in the general case so, we can conclude that F is not a homomorphism.

On the other hand, $fg(t) = t_1 \cdot t_2 = P(t)$ — homogeneous polynomial of second degree vector variable t. It is known that $P(t) = B(t,t)$ is bilinear form which is uniquely determined by the polarization formula:

$$B(t,t) = \frac{t_1 t_2 + t_2 t_1}{2}.$$

Then

$$\bar{B}(x,x) = \frac{x_1 * x_2 + x_2 * x_1}{2},$$

and we have

$$\tilde{\bar{B}}(u) = \frac{u_1 \Box u_2 + u_1 \Diamond u_2}{2} = \frac{u_2 \Box u_1 + u_2 \Diamond u_1}{2}.$$

So, $\tilde{\bar{B}}(u,u) = \tilde{\bar{P}}(u) = \widetilde{\overline{fg}}(t) \neq \tilde{\bar{f}}(t)\tilde{\bar{g}}(t)$.

Next, we consider the case when A is a reflexive Banach algebra. Let us denote by $\mathcal{P}(^nX)$ the Banach space of all continuous n-homogeneous complex-valued polynomials on X. $\mathcal{P}_f(^nX)$ denotes the subspace of n-homogeneous polynomials of finite type, that is, the subspace generated by finite sum of finite products of linear continuous functionals. The closure of $\mathcal{P}_f(^nX)$ in the topology of uniform convergence on bounded sets is called the space of approximable polynomials and denoted by $\mathcal{P}_c(^nX)$.

Let us denote by $A_n(X)$ the closure of the algebra, generated by polynomials from $\mathcal{P}(^{\leq n}X)$ with respect to the uniform topology on bounded subsets of X. It is clear that $A_1(X) \cap \mathcal{P}(^nX) = \mathcal{P}_c(^nX)$.

Let us denote by $\mathcal{L}(H_b(X), A)$ the space of all continuous n-linear operators on $H_b(X)$ to A and let $M_A(H_b(X))$ be the set of all homomorphisms on $H_b(X)$ to A.

In [4] introduced a concept of radius function $R(\varphi)$ of a given linear functional $\varphi \in H_b(X)'$ as the infimum of all numbers $r > 0$ such that φ is bounded with respect to the norm of uniform convergence on the ball rB and proved that

$$R(\varphi) = \limsup_{n \to \infty} \|\varphi_n\|^{1/n},$$

where φ_n is the restriction of φ to $\mathcal{P}(^nX)$. In [7] extended this definition to a homomorphism $\Phi \in M_A(H_b(X))$, that is, $R(\Phi)$ is the infimum of all numbers $r > 0$ such that Φ is bounded with respect to the norm of uniform convergence on the ball rB and proved that

$$R(\Phi) = \limsup_{n \to \infty} \|\Phi_n\|^{1/n}, \tag{1}$$

where Φ_n is the restriction of Φ to space n-homogeneous polynomials.

Theorem 1. *Suppose that $\Phi_n \in \mathcal{L}(\mathcal{P}(^nX), A)$ for $n \in \mathbb{Z}_+$, and suppose that the norms of Φ_n on $\mathcal{P}(^nX)$ satisfy*

$$\|\Phi_n\| \leq cs^n$$

for $c, s > 0$. Then there is a unique $\Phi \in \mathcal{L}(H_b(X), A)$ whose restriction to $\mathcal{P}(^nX)$ coincides with Φ_n for every $n \in \mathbb{Z}_+$.

Proof. For any character $\theta \in M(A)$, $\|\theta\| = 1$ we construct operator $\Phi_n : \mathcal{P}(^nX) \to A$. Then $\theta \circ \Phi_n \in (\mathcal{P}(^nX))'$ and $\|\theta \circ \Phi_n\| \leq \|\Phi_n\|$. Since $\|\Phi_n\| \leq cs^n$, then every θ satisfies the inequality $\|\theta \circ \Phi_n\| \leq cs^n$. From [4, Proposition 2.4] it follows that for every θ there exists linear functional $\varphi : H_b(X) \to \mathbb{C}$, $\varphi \in H_b(X)'$, such that $\varphi_n = \theta \circ \Phi_n$. Therefore, we have operator $T : A' \to H_b(X)'$, $\theta \mapsto \varphi$ and T^* is the adjoint operator to T:

$$T^* : H_b(X)'' \to A'' = A.$$

Let us consider the restriction of T^* on $H_b(X) \subset H_b(X)''$ and denoted it by Φ. Clearly $\Phi : H_b(X) \to A$ is a required operator.

In order to prove that the restriction Φ to $\mathcal{P}(^nX)$ coincides with Φ_n it is enough to show that $\Phi_n(P) = \Phi(P)$ for every $P \in \mathcal{P}(^nX)$. Put $\Phi_n(P) = a_1$, $\theta(a_1) = c_1 \in \mathbb{C}$, that is $(\theta \circ \Phi_n)(P) = \varphi_n(P) = c_1$. On the other hand, $\Phi(P) = a_2$, that is $(\theta \circ \Phi)(P) = \varphi(P) = c_2$. Since φ_n is restriction of φ, $\varphi(P) = c_2 = \varphi_n(P) = c_1$, $\Rightarrow c_1 = c_2 = c$. So, the equality $(\theta \circ \Phi)(P) = (\theta \circ \Phi_n)(P) = c$ for every θ implies that $\Phi_n(P) = \Phi(P)$. □

In the work [6] it was formulated and proved the Lemma 1 on extension of the linear functional $\varphi \in H_b(X)'$ to character $\psi \in M_b$. The following theorem is a generalization of the known lemma and is related to the study of extension of linear operator to the homomorphism.

Theorem 2. *Let $\Phi \in \mathcal{L}(H_b(A \otimes_\pi X), A)$ be a linear operator such that $\Phi(P) = 0$ for every $P \in \mathcal{P}(^m(A \otimes_\pi X), A) \cap A_{m-1}(A \otimes_\pi X)$, where m is a fixed positive integer and Φ_m be the nonzero restriction of Φ to $\mathcal{P}(^m(A \otimes_\pi X))$.*

Then there is a homomorphism $\Psi \in M_A(H_b(A \otimes_\pi X))$ such that its restrictions Ψ_k to $\mathcal{P}(^k(A \otimes_\pi X))$ satisfy the conditions: $\Psi_k = 0$ for all $k < m$ and $\Psi_m = \Phi_m$. Moreover, the radius functions of Ψ is calculated by the formula

$$\|\Phi_m\|^{1/m} \leq R(\Psi) \leq e\|\Phi_m\|^{1/m}.$$

Proof. For every polynomial $P \in \mathcal{P}(^{mk}(A \otimes_\pi X))$ we denote by $P_{(m)}$ the polynomial from $\mathcal{P}(^k \otimes^m_{s,\pi}(A \otimes_\pi X))$ such that $P_{(m)}(\bar{a}^{\otimes m}) = P(\bar{a})$.

Since $\Phi_m \neq 0$, there is an element $\omega \in (A \otimes_{s,\pi} X)''$, $\omega \neq 0$ such that for any m-homogeneous polynomial P,

$$\Phi(P) = \Phi_m(P) = \widetilde{P}_{(m)}(w), \qquad \|w\| = \|\Phi_m\|,$$

where $\widetilde{P}_{(m)}$ is the Aron-Berner extension of linear functional $P_{(m)}$ from $\otimes^m_{s,\pi}(A \otimes_\pi X)$ to $\otimes^m_{s,\pi}(A \otimes_\pi X)''$. For an arbitrary n-homogeneous polynomial $Q \in \mathcal{P}(^n(A \otimes_\pi X))$ we set

$$\Psi(Q) = \begin{cases} \widetilde{Q}_{(m)}(w) & \text{if } n = mk \text{ for some } k \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $\widetilde{Q}_{(m)}$ is the Aron-Berner extension of the k-homogeneous polynomial $Q_{(m)}$ from $\otimes^m_{s,\pi}(A \otimes_\pi X)$ to $\otimes^m_{s,\pi}(A \otimes_\pi X)''$.

Let (u_α) be a net from $\otimes^m_{s,\pi}(A \otimes_\pi X)$ which converges to w in the weak-star topology of $\otimes^m_{s,\pi}(A \otimes_\pi X)''$, where α belongs to an index set $\mathfrak{A}$. We can assume that every u_α has a representation $u_\alpha = \sum_{j \in \mathbb{N}} (a_{j,\alpha} \otimes_\pi x_{j,\alpha})^{\otimes m} = \sum_{j \in \mathbb{N}} p_{j,\alpha}^{\otimes m}$ for some $a_{j,\alpha} \in A, x_{j,\alpha} \in X$.

Now we will show that $\Psi(PQ) = \Psi(P)\Psi(Q)$ for any homogeneous polynomials P and Q.

1) Let us suppose first that $\deg(PQ) = mr + l$ for some integers $r \geq 0$ and $m > l > 0$. Then P or Q has degree equal to $mk + s$, $k \geq 0$, $m > s > 0$. Thus, by the definition $\Psi(PQ) = 0$ and $\Psi(P)\Psi(Q) = 0$.

2) Suppose now that for some integer $r \geq 0$ $\deg(PQ) = mr$. If $\deg P = mk$ and $\deg Q = mn$ for $k, n \geq 0$, then $\deg(PQ) = m(k+n)$ and

$$\Psi(PQ) = (\widetilde{PQ})_{(m)}(w) = \widetilde{P}_{(m)}(w)\widetilde{Q}_{(m)}(w) = \Psi(P)\Psi(Q).$$

3)Let at last $\deg P = mk + l$ and $\deg Q = mn + r$, $l, r > 0$, $l + r = m$. Write

$$\nu = \frac{1}{(\deg P + \deg Q)!} = \frac{1}{(m(k+n+1))!}.$$

Denote by F_{PQ} the symmetric multilinear map, associated with PQ. Then

$$F_{PQ}\left(\bar{a}_1, \ldots, \bar{a}_{m(k+n+1)}\right) = \nu \sum_{\sigma \in \mathfrak{S}_{m(k+n+1)}} F_P\left(\bar{a}_{\sigma(1)}, \ldots, \bar{a}_{\sigma(mk+l)}\right) F_Q\left(\bar{a}_{\sigma(mk+l+1)}, \ldots, \bar{a}_{\sigma(m(k+n+1))}\right),$$

where $\mathfrak{S}_{m(k+n+1)}$ is the group of permutations on $\{1, \ldots, m(k+n+1)\}$. Thus, for $\alpha_1, \ldots, \alpha_{k+n+1} \in \mathfrak{A}$ we have

$$\begin{aligned}
\psi(PQ) = (\widetilde{PQ})_{(m)}(w) &= \lim_{\alpha_1,\ldots,\alpha_{k+n+1}} \widetilde{F}_{PQ_{(m)}}\left(u_{\alpha_1}, \ldots, u_{\alpha_{k+n+1}}\right) \\
&= \lim_{\alpha_1,\ldots,\alpha_{k+n+1}} \widetilde{F}_{PQ_{(m)}}\left(\sum_{j\in\mathbb{N}} p_{j,\alpha_1}^{\otimes m}, \ldots, \sum_{j\in\mathbb{N}} p_{j,\alpha_{k+n+1}}^{\otimes m}\right) \\
&= \nu \sum_{\sigma \in \mathfrak{S}_{m(k+n+1)}} \lim_{\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+n+1)}} \\
&\quad \sum_{j_1,\ldots,j_{k+n+1}\in\mathbb{N}} F_P\left(\bar{a}^m_{j_{\sigma(1)},\alpha_{\sigma(1)}}, \ldots, \bar{a}^m_{j_{\sigma(k)},\alpha_{\sigma(k)}}, \bar{a}^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}\right) \\
&\quad \times F_Q\left(\bar{a}^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}, \bar{a}^m_{j_{\sigma(k+2)},\alpha_{\sigma(k+2)}}, \ldots, \bar{a}^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}\right).
\end{aligned}$$

Fix some $\sigma \in \mathfrak{S}_{m(k+n+1)}$ and fix all $\bar{a}_{j_{\sigma(i)},\alpha_{\sigma(i)}}$ for $i \leq k$ and for $i > k+1$. Then

$$\begin{aligned}
\sum_{j_1,\ldots,j_{k+n+1}\in\mathbb{N}} \lim_{\alpha_{\sigma(k+1)}} & F_P\left(\bar{a}^m_{j_{\sigma(1)},\alpha_{\sigma(1)}}, \ldots, \bar{a}^m_{j_{\sigma(k)},\alpha_{\sigma(k)}}, \bar{a}^l_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}\right) \\
& \times F_Q\left(\bar{a}^r_{j_{\sigma(k+1)},\alpha_{\sigma(k+1)}}, \bar{a}^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}}, \ldots, \bar{a}^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}\right) = 0,
\end{aligned}$$

because for a fixed $\bar{a}_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i \leq k$,

$$P_\sigma(y) := \sum_{j_1,\ldots,j_k,j_{k+2},\ldots,j_{k+n+1}\in\mathbb{N}} F_P\left(\bar{a}^m_{j_{\sigma(1)},\alpha_{\sigma(1)}}, \ldots, \bar{a}^m_{j_{\sigma(k)},\alpha_{\sigma(k)}}, y^l\right)$$

is an l-homogeneous polynomial and for fixed $\bar{a}_{k_{\sigma(i)},\alpha_{\sigma(i)}}$, $i > k+1$,

$$Q_\sigma(y) := \sum_{j_1,\ldots,j_k,j_{k+2},\ldots,j_{k+n+1}\in\mathbb{N}} F_Q\left(y^r, \bar{a}^m_{j_{\sigma(k+2)},\alpha_{\sigma(n+2)}}, \ldots, \bar{a}^m_{j_{\sigma(k+n+1)},\alpha_{\sigma(k+n+1)}}\right)$$

is an r-homogeneous polynomial. Thus, $P_\sigma Q_\sigma \in \mathcal{A}_{m-1}(A \otimes_\pi X)$. Hence,

$$\lim_\alpha (P_\sigma Q_\sigma)_{(m)}(u_\alpha) = \Psi(P_\sigma Q_\sigma) = 0$$

for every fixed σ. Therefore, $\Psi(PQ) = 0$. On the other hand, $\Psi(P)\Psi(Q) = 0$ by the definition of Ψ. So, $\Psi(PQ) = \Psi(P)\Psi(Q)$.

Thus, we have defined the multiplicative operator Ψ on homogeneous polynomials. We can extend it by linearity and distributivity to a homomorphism on the algebra of all continuous polynomials $\mathcal{P}(A \otimes_\pi X)$.

If Ψ_n is the restriction of Ψ to $\mathcal{P}(^n(A \otimes_\pi X))$, then $\|\Psi_n\| = \|w\|^{n/m}$ if n/m is a positive integer and $\|\Psi_n\| = 0$ otherwise. Hence, the series

$$\Psi = \sum_{n\in\mathbb{N}} \Psi_n$$

is a continuous homomorphism on $H_b(A \otimes_\pi X)$ by Theorem 1 and the radius function of Ψ can be computed by $R(\Psi) = \limsup_{n\to\infty} \|\Psi_n\|^{1/n} \geq \limsup_{n\to\infty} \|w\|^{n/mn} = \|w\|^{1/m} = \|\Phi_m\|^{1/m}$. On the other hand, $\|\Psi_n\| = \sup_{\|P\|=1} |\Psi_n(P)| = \sup_{\|P\|=1} |P_{(m)}(w)|$. Since

$$|P_{(m)}(w)| \leq \|w\|^{n/m} \|P_{(m)}\| \leq c(n, A \otimes_\pi X) \|w\|^{n/m} \|P\|,$$

we have

$$\|\Psi_n\| \leq c(n, A \otimes_\pi X) \|w\|^{n/m} \leq \frac{n^n}{n!} \|w\|^{n/m} = \frac{n^n}{n!} \|\Phi_m\|^{n/m}.$$

So $R(\Psi) \leq e\|\Phi_m\|^{1/m}$. The theorem is proved. □

REFERENCES

[1] Arens R. *The adjoint of a bilinear operation*. Proc. Amer. Math. Soc. 1951, **2**, 839–848. doi:10.1090/S0002-9939-1951-0045941-1

[2] Daws M. *Arens regularity of the algebra of operators on a Banach space*. Bull. London. Math. Soc. 2004, **36** (4), 493–503. doi:10.1112/S0024609303003072

[3] Cabello S.F., Garcia R., Villanueva I. *Extension of multilinear operators on Banach spaces*. Extracta Math. 2000, **15** (2), 291–334.

[4] Aron R.M., Cole B.J., Gamelin T.W. *Spectra of algebras of analytic functions on a Banach space*. J. Reine Angew. Math. 1991, **415**, 51–93.

[5] Dineen S., Hart R.E., Taylor C. *Spectra of tensor product elements III : holomorphic properties*. Math. Proc. R. Ir. Acad. 2003, **103** (1), 61–92.

[6] Zagorodnyuk A.V. *Spectra of algebras of entire functions on Banach spaces*. Proc. Amer. Math. Soc.2006, **134**, 2559–2569. doi:10.1090/S0002-9939-06-08260-8.

[7] Zagorodnyuk A.V., Petriv H.M. *Homomorphisms of algebras of entire functions of bounded type on a Banach space*. Appl. Probl. Mech. Math. 2013, **11**, 7–11. (in Ukrainian)

Received 25.03.2015

Приймак Г.М. *Гомоморфізми і функціональне числення в алгебрах цілих функцій на банахових просторах* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 108–113.

Досліджено гомоморфізми алгебри цілих функцій обмеженого типу на банахових просторах в комутативну банахову алгебру. Зокрема, запропоновано метод побудови гомоморфізмів, які є нулем на однорідних поліномах степеня, що не перевищує деяке фіксоване число n.

Ключові слова і фрази: Продовження Арона-Бернера, функціональне числення, алгебри аналітичних функцій в банахових просторах.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 114–119
doi:10.15330/cmp.7.1.114-119

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.114–119

TROSHKI V.B.

A NEW CRITERION OF TESTING HYPOTHESIS ABOUT THE COVARIANCE FUNCTION OF THE HOMOGENEOUS AND ISOTROPIC RANDOM FIELD

In this paper we consider a continuous in mean square homogeneous and isotropic Gaussian random field. A criterion for testing hypotheses about the covariance function of such field using estimates for its norm in the space $L_p(\mathbb{T}), p \geq 1$, is constructed.

Key words and phrases: criterion for testing hypotheses, spherical correlogram, isotropic field.

Taras Shevchenko National University, 64/13 Volodymyrska str., 01601, Kyiv, Ukraine
E-mail: btroshki@ukr.net

INTRODUCTION

Since the majority of the papers is devoted to the evaluation of covariance function with given accuracy in the uniform metric that is why in this paper we set the task to estimate the covariance function $B(\tau)$ of a Gaussian homogeneous isotropic random field with given accuracy and reliability in $L_p(T), p \geq 1$. We construct a criterion for testing the hypothesis that the covariance function of homogeneous and isotropic Gaussian random field $\xi(x)$ equals $B(\tau)$. We shall use spherical correlogram

$$\hat{B}(\tau) = \frac{1}{U_n(R)} \int_{V_R(0)} \xi(x)\eta_\tau(x)dx$$

as the estimator of the function $B(\tau)$.

Definition of the square Gaussian random vector was introduced by Yu. Kozachenko and O. Moklyachuk in the paper [9]. They also received estimates for distributions of square Gaussian random vectors. Applications of the theory of square Gaussian random variables and stochastic processes in mathematical statistics were considered in the paper [8] and in the book [3]. A lot of the papers so far have been dedicated to estimation of covariance function of Gaussian random process and field, in particular the books [5], [1] and [15]. The main properties of the correlograms of the stationary Gaussian stochastic processes were studied by V. Buldygin and Yu. Kozachenko in the book [3]. Exponential inequalities for the distribution of the deviations correlograms from respective covariance function in the uniform metric were considered in the papers [8], [10] and [11]. Asymptotic normality of correlograms in the space of continuous functions were given by V. Buldygin and V. Zayats in the paper [4]. Issues of asymptotic normality of correlograms in the certain functional spaces were discussed in the papers by O. Ivanov [6] and V. Buldygin [2]. Leonenko and O. Ivanov in the book [7] considered asymptotic properties for estimates of covariance functions. In the papers [14] and [13]

УДК 519.21
2010 *Mathematics Subject Classification:* 60G17, 60G07.

Yu. Kozachenko and T. Fedoryanich constructed a criterion for testing hypotheses about the covariance function of a Gaussian stationary process. A criterion for testing hypotheses about the covariance function of a stationary Gaussian stochastic process with given accuracy and reliability in $L_p(\mathbb{T}), p \geq 1$ is constructed in the paper [12].

1 REQUIRED INFORMATION

Definition 1 ([3]). *Let $\mathbb{T}$ be a parametric set and let $\Xi = \{\xi_t : t \in \mathbf{T}\}$ be a family of Gaussian random variables such that $\mathbf{E}\xi_t = 0$. The space $SG_\Xi(\Omega)$ is called a space of square Gaussian random variables if any $\zeta \in SG_\Xi(\Omega)$ can be represented as*

$$\zeta = \bar{\xi}^T A \bar{\xi} - \mathbf{E}\bar{\xi}^T A \bar{\xi},$$

where $\bar{\xi} = (\xi_1, ..., \xi_N)^T$ with $\xi_k \in \Xi, k = 1, ..., n$, and A is an arbitrary matrix with real-valued entries, or if $\zeta \in SG_\Xi(\Omega)$ has the representation

$$\zeta = \lim_{n\to\infty} \left(\bar{\xi}_n^T A \bar{\xi}_n - \mathbf{E}\bar{\xi}_n^T A \bar{\xi}_n \right).$$

Theorem 1 ([12]). *Let $\{\mathbb{T}, \mathfrak{A}, \mu\}$be a measurable space, where $\mathbb{T}$ is a parametric set and let $X = \{X(t), t \in \mathbb{T}\}$ be a square Gaussian stochastic process. Suppose that X is a measurable process. Further, let the Lebesgue integral $\int\limits_{\mathbb{T}} (\mathbf{E}X^2(t))^{\frac{p}{2}} d\mu(t)$ be well defined for $p \geq 1$. Then the integral $\int\limits_{\mathbb{T}} (X(t))^p d\mu(t)$ exists with probability 1 and*

$$P\left\{ \int\limits_{\mathbb{T}} | X(t) |^p \, d\mu(t) > \varepsilon \right\} \leq 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}} \right\} \tag{1}$$

for all $\varepsilon \geq \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$, where $C_p = \int\limits_{\mathbb{T}} (\mathbf{E}X^2(t))^{\frac{p}{2}} d\mu(t)$.

Definition 2 ([15]). *Random field $\xi = \{\xi(x), t \in \mathbb{R}^n\}$ is called homogeneous in the wide sense in $\mathbb{R}^n$ if $\mathbb{E}\xi(x) = const, x \in \mathbb{R}^n$ and*

$$\mathbb{E}\xi(x)\xi(y) = B(x-y) = \int\limits_{\mathbb{R}^n} e^{i(\lambda, x-y)} dF(\lambda), x, y \in \mathbb{R}^n.$$

Definition 3 ([15]). *Let $SO(n)$ be a group of rotations $\mathbb{R}^n$ around the origin. Homogeneous random field $\xi(x)$ is called isotropic if $\mathbf{E}\xi(x)\overline{\xi(y)} = \mathbf{E}\xi(gx)\overline{\xi(gy)}$ for all $g \in SO(n)$.*

We denote by $S_R(x)$ and $V_R(x)$ sphere and ball of radius R centered at a point x respectively. Let $m_n^{(R)}(\cdot)$ be a Lebesgue measure on $S_R(x)$. By $U_n(R)$ and $\omega_n(R)$ we denote the volume of ball and the surface area of the sphere of radius R in $\mathbb{R}^n$ respectively.

Consider a random field

$$\eta_R(x) = \frac{1}{\omega_n(R)} \int_{S_R(x)} \xi(y) m_n^{(R)}(dy).$$

Theorem 2 ([15]). *Random field $\eta_R(x)$ is homogeneous and isotropic. Homogeneous and isotropic random fields $\eta_R(x)$ and $\xi(x)$ are related each other and the following equalities hold*

$$\mathbf{E}\eta_{R_1}(x_1)\eta_{R_2}(x_2) = \int_0^\infty Y_n(\lambda R_1)Y_n(\lambda R_2)Y_n(\lambda\tau_{x_1x_2})d\Phi(\lambda), \tag{2}$$

$$\mathbf{E}\eta_{R_1}(x_1)\xi(x_2) = \int_0^\infty Y_n(\lambda R)Y_n(\lambda\tau_{x_1x_2})d\Phi(\lambda), \tag{3}$$

where

- $Y_n(z) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(z)}{z^{\frac{n-2}{2}}}$ *is a spherical Bessel function,* $\Phi(\lambda) = \int\limits_{\sqrt{v_1^2+\dots+v_n^2}\le\lambda} F(dv)$, $F(\cdot)$ *is a finite measure on σ-algebra B_n Borel sets of $\mathbb{R}^n$.*
- $\tau_{x_1x_2} = |x_1 - x_2|$ *is a distance between the points x_1 and x_2.*

2 Construction criterion for testing hypothesis about the covariance function of the homogeneous and isotropic random field

Let $\xi(x)$ be a continuous in mean square homogeneous and isotropic Gaussian random field in $\mathbb{R}^n$ with zero-mean. Without any loss of generality, we can assume that the sample paths of the field $\xi(x)$ are continuous with probability one on any bounded and closed set.

Let the random field $\xi(x)$ be observed on the ball $V_{R+\tau}(0)$, $\tau \ge 0$, and let the spectral function of the field $\Phi(\lambda)$ be absolutely continuous.

Theorem 3. *Let a spherical correlogram*

$$\hat{B}(\tau) = \frac{1}{U_n(R)}\int_{V_R(0)}\xi(x)\left(\frac{1}{\omega_n(r)}\int_{S_r(x)}\xi(t)m_n^{(\tau)}(dt)\right)dx = \frac{1}{U_n(R)}\int_{V_R(0)}\xi(x)\eta_\tau(x)dx \tag{4}$$

be an estimator of the covariance function $B(\tau)$. Then the following inequality holds for all $\varepsilon \ge \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$:

$$P\left\{\int_0^A(\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon\right\} \le 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\},$$

where

$$C_p = \frac{1}{U_n^2(R)}\int_0^A\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) + \left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdyd\tau$$

and $0 < A < \infty$.

Remark 1. *Since the sample paths of the field $\xi(x)$ are continuous with probability one on the ball $V_{R+\tau}(0)$, $\hat{B}(\tau)$ is a Riemann integral.*

Proof. Consider

$$\mathbf{E}(\hat{B}(\tau) - B(\tau))^2 = \mathbf{E}(\hat{B}(\tau))^2 - B^2(\tau).$$

From the Isserlis equality for jointly Gaussian random variables and relationships (2) and (3) it follows that

$$\begin{aligned}
\mathbf{E}\hat{B}^2(\tau) &= \frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}(\mathbf{E}\xi(x)\eta_\tau(x)\mathbf{E}\xi(x)\eta_\tau(x) \\
&+ \mathbf{E}\xi(x)\xi(y)\mathbf{E}\eta_\tau(x)\eta_\tau(y) + \mathbf{E}\xi(x)\eta_\tau(y)\mathbf{E}\xi(y)\eta_\tau(x))\,dxdy \\
&= \frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(\left[\int_0^\infty Y_n(\lambda\tau)Y_n(0)d\Phi(\lambda)\right]^2\right. \\
&+ B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) \\
&+ \left.\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right)dxdy \\
&= \frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(B^2(\tau) + B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right. \\
&+ \left.\left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]^2\right)dxdy = B^2(\tau) \\
&+ \frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right. \\
&+ \left.\left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdy.
\end{aligned}$$

Therefore,

$$\mathbf{E}(\hat{B}(\tau) - B(\tau))^2 = \frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) + \left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdy. \tag{5}$$

Since $\hat{B}(\tau) - B(\tau)$ is a square Gaussian random field (see Lemma 3.1, Chapter 6 in book [3]), then it follows from the Theorem 1 that

$$P\left\{\int_0^A(\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon\right\} \le 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

Applying equality (5) we get

$$C_p = \frac{1}{U_n^2(R)}\int_0^A\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) + \left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdyd\tau.$$

□

Denote

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}} \right\}.$$

From the Theorem 3 it follows that if $\varepsilon \geq z_p = C_p \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p} \right)^p$, then

$$P\left\{ \int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon \right\} \leq g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta, 0 < \delta < 1$. Put $S_\delta = \max\{\varepsilon_\delta, z_p\}$. It is obviously that $g(S_\delta) \leq \delta$ and

$$P\left\{ \int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > S_\delta \right\} \leq \delta. \tag{6}$$

Let $\mathbb{H}$ be the hypothesis that the covariance function of homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ equals $B(\tau)$ for $0 \leq \tau \leq A$. From the Theorem 3 and (6) it follows that to test the hypothesis $\mathbb{H}$ one can use the following criterion.

Criterion 1 For a given level of confidence δ the hypothesis $\mathbb{H}$ is accepted if

$$\int_0^A (\hat{B}(\tau) - B(\tau))^p d\mu(\tau) < S_\delta$$

otherwise hypothesis is rejected.

Remark 2. *The equation $g(\varepsilon) = \delta$ has a solution for any $\delta > 0$, since $g(\varepsilon)$ is a monotonically decreasing function. We can find the solution of equation using numerical methods.*

Remark 3. *One can easily see that Criterion 1 can be used if $C_p \to 0$ as $R \to \infty$.*

3 CONCLUSIONS

In this paper, we constructed a new criterion for testing hypothesis about the covariance function of homogeneous and isotropic Gaussian random field. The evaluation is carried out by observing for the random field on the ball. We regard spherical correlogram as the estimator of the covariance function.

REFERENCES

[1] Bendat J., Piersol A. Applications of Correlation and Spectral Analysis. John Wiley & Sons, New York, 1980.

[2] Buldygin V. *On the properties of the empirical correlogram of Gaussian process with integrable in squared spectral density.* Ukrainian Math. J. 1995, **47** (7), 1006–1021. doi:10.1007/BF01084897 (translation of Ukrain. Mat. Zh. 1995, **47** (7), 876–889. (in Russian))

[3] Buldygin V., Kozachenko Yu. Metric characterization of random variables and random processes. Amer. Math. Soc., Providence, RI, 2000.

[4] Buldygin V., Zayats V. *On the asymptotic normality of estimates of the correlation functions stationary Gaussian processes in spaces of continuous functions.* Ukrainian Math. J. 1995, **47** (11), 1696–1710. doi:10.1007/BF01057918 (translation of Ukrain. Mat. Zh. 1995, **47** (11), 1485–1497. (in Russian))

[5] Jenkins G., Watts D. Spectral Analysis and Its Applications. Holden Day, Merrifield, 1971.

[6] Ivanov A. *A limit theorem for the evaluation of the correlation function.* Theory Prob. Math. Statist. 1978, **19**, 76–81

[7] Ivanov A., Leonenko N. Statistical Analysis of Random Fields. Kluwer, Dordrecht, 1989.

[8] Kozachenko Yu.V., Kozachenko L.F. *A test for a hypothesis on the correlation function of Gaussian random process.* J. Math. Sci. (New York) 1995, **77** (5), 3437–3444. doi:10.1007/BF02367991 (translation of J. Obchysl. Prykl. Mat. 1993, **77**, 61–74. (in Ukrainian))

[9] Kozachenko Yu., Moklyachuk O. *Pre-Gaussian random vectors and their application.* Theory Prob. Math. Statist. 1994, **50**, 87–96.

[10] Kozachenko Yu., Oleshko T. *Analytic properties of certain classes of pre-Gaussian stochastic processes.* Theor. Prob. Math. Statist. 1993, **48**, 37–51.

[11] Kozachenko Yu., Stadnik A. *On the convergence of some functionals of Gaussian vectors in Orlicz spaces.* Theor. Probab. Math. Statist. 1991, **44**, 80–87.

[12] Kozachenko Yu., Troshki V. *A criterion for testing hypotheses about the covariance function of a stationary Gaussian stochastic process.* Modern Stoch.: Theory and Appl. 2014, **1**, 139–149. doi:10.15559/15-VMSTA17

[13] Kozachenko Yu., Fedoryanych T. *A criterion for testing hypotheses about the covarians function of a Gaussian stationary process.* Theor. Probab. Math. Statist. 2005, **69**, 85–94.

[14] Fedoryanych T. *One estimate of the correlation function for Gaussian stochastic process.* Bull. Taras Shevchenko National University of Kyiv 2004, **11**, 72–76. (in Ukrainian)

[15] Yadrenko M. Spectral Theory of Random Fields. Vyshcha Shkola, Kyiv, 1980. (in Russian)

Received 10.02.2015

Трошкі В.Б. *Новий критерій для перевірки гіпотези про вигляд коваріаційної функції однорідного та ізотропного випадкового поля* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 114–119.

В даній роботі розглядаються однорідне та ізотропне неперервне в середньому квадратичному випадкове поле. Тут побудований критерій для перевірки гіпотези про вигляд коваріаційної функції однорідного та ізотропного випадкового поля.

Ключові слова і фрази: критерій для перевірки гіпотези, сферична корелограма, ізотропне поле.

ISSN 2075-9827
Carpathian Math. Publ. 2015, 7 (1), 120–132
doi:10.15330/cmp.7.1.120-132

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №1, С.120–132

CHUPORDIA V.A.

ON THE STRUCTURE OF SOME MINIMAX-ANTIFINITARY MODULES

Let R be a ring and G be a group. An R-module A is said to be *minimax* if A includes a noetherian submodule B such that A/B is artinian. It is studied a $\mathbb{Z}_{p^\infty}G$-module A such that $A/C_A(H)$ is minimax as a $\mathbb{Z}_{p^\infty}$-module for every proper not finitely generated subgroup H.

Key words and phrases: minimax module, cocentralizer, module over group ring, minimax-antifinitary RG-module, generalized radical group.

Oles Honchar Dnipropetrovsk National University, 72 Gagarin avenue, 49010, Dnipropetrovsk, Ukraine
E-mail: vchupordya@mail.ru

INTRODUCTION

The modules over group rings RG are classical objects of study with well established links to various areas of algebra. The case when G is a finite group has been studying in sufficient details for a long time. For the case when G is an infinite group, the situation is different. The investigation of modules over polycyclic-by-finite groups was initiated in the classical works of P. Hall [3, 4]. Nowadays, the theory of modules over polycyclic-by-finite groups is highly developed and rich on interesting results. This was largely due to the fact that a group ring RG of a polycyclic-by-finite group G over a noetherian ring R is also noetherian. This allowed developing an advanced theory of such rings and obtain deep results about their structure. For group rings over some other groups (even over well-studied groups, as for instance, the Chernikov groups) the situation is not so good since these rings have quite a sophisticated structure. In particular, they are neither Noetherian nor Artinian. In such cases, it is not always possible to conduct the study of modules based only on the ring properties. So naturally there is a need for other approaches. Application of the finiteness conditions, particularly the use of the minimal and maximal conditions, proved to be very effective in the classical theory of rings and modules. Noetherian and artinian modules over group rings are also very well investigated. Many aspects of the theory of artinian modules over group rings are well reflected in the book [9]. Lately the so-called finitary approach is under intensive development. This is mainly due to the progress which its applications have found in the theory of infinite dimensional linear groups.

Let R be a ring, G a group and A an RG-module. For a subgroup H of G we consider the R- submodule $C_A(H)$. Then H acts on $A/C_A(H)$. The R-factor-module $A/C_A(H)$ is called the *cocentralizer of H in A*. The factor-group $H/C_H(A/C_A(H))$ is isomorphic to a subgroup of automorphisms group of an R-module $A/C_A(H)$. If x is an element of $C_H(A/C_A(H))$, then x acts trivially on factors of the series $\langle 0\rangle \le C_A(H) \le A$. It follows that $C_H(A/C_A(H))$

УДК 512.544
2010 *Mathematics Subject Classification:* 20C07, 20F19.

is abelian. This shows that the structure of H to a greater extent is defined by the structure of $C_H(A/C_A(H))$, and hence by the structure of the automorphisms group of the R-module $A/C_A(H)$.

Let $\mathfrak{M}$ be a class of R-modules. We say that A is an *$\mathfrak{M}$-finitary module over RG* if $A/C_A(x) \in \mathfrak{M}$ for each element $x \in G$. If R is a field, $C_G(A) = \langle 1\rangle$, and $\mathfrak{M}$ is the class of all finite dimensional vector spaces over R, then we come to the finitary linear groups. The theory of finitary linear groups is quite well developed (see, for example, the survey [11]). B.A.F. Wehrfritz began considering the cases when $\mathfrak{M}$ is the class of finite R-modules [13, 15, 16, 18], when $\mathfrak{M}$ is the class of noetherian R-modules [14], when $\mathfrak{M}$ is the class of artinian R- modules [16–20]. The artinian-finitary modules have been considered also in the paper [10]. The artinian and noetherian modules can be united into the following type of modules. An R-module A is said to be *minimax* if A has a finite series of submodules, whose factors are either noetherian or artinian. It is not hard to show that if R is an integral domain, then every minimax R-module A includes a noetherian submodule B such that A/B is artinian. The first natural case here is the case when $R = \mathbb{Z}$ is the ring of all integers. B.A.F. Wehrfritz has began the study of noetherian-finitary and artinian-finitary modules with separate consideration of this case. This case is of particular importance in applications, for instance, it is very important in the theory of generalized soluble groups.

Let G be a group, A an RG-module, and $\mathfrak{M}$ a class of R-modules. Put

$$\mathcal{C}_{\mathfrak{M}}(G) = \{H \mid H \text{ is a subgroup of } G \text{ such that } A/C_A(H) \in \mathfrak{M}\}.$$

If A is an $\mathfrak{M}$-finitary module, then $\mathcal{C}_{\mathfrak{M}}(G)$ contains every cyclic subgroup (moreover, every finitely generated subgroup whenever $\mathfrak{M}$ satisfies some natural restrictions). It is clear that the structure of G depends significantly on which important subfamilies of the family $\Lambda(G)$ of all proper subgroups of G include $\mathcal{C}_{\mathfrak{M}}(G)$. Therefore it is interesting to consider the cases when the family $\mathcal{C}_{\mathfrak{M}}(G)$ is large. In almost all groups (with exception of noetherian groups), the family of subgroups which is not finitely generated is much larger than the family of finitely generated subgroups. It is therefore interesting to consider the case, which is dual to the case of an $\mathfrak{M}$-finitary module.

Let R be a ring, G be a group and A be an RG-module. We say that A is *minimax-antifinitary RG-module* if the factor-module $A/C_A(H)$ is minimax as an R-module for each not finitely generated proper subgroup H and the R-module $A/C_A(G)$ is not minimax.

This current work is devoted to the study of the minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-modules. Here $\mathbb{Z}_{p^\infty}$ denotes a ring of p-adic number. The ring $\mathbb{Z}_{p^\infty}$ play a very specific role in the theory of modules over group rings. It is principal ideal domain and, in the other hand, it is a valuation ring. The study breaks down naturally into the following cases. Put

$$\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx(G)} = \{x \mid A/C_A(x) \text{ is a minimax } \mathbb{Z}_{p^\infty} - \text{module}\}.$$

The first case is the case when $G = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx(G)}$. In this case, every proper subgroup of G has a minimax cocentralizer. This case was considered separately in another paper. The second case is the case when $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ and the group G is not finitely generated. The third case is the case when $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ and the group G is finitely generated. The current article is dedicated to the second case. Here we consider the modules over groups, which belong to the following very large class of groups.

A group G is called *generalized radical*, if G has an ascending series whose factors are locally nilpotent or locally finite. Hence a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical $\mathbf{Lnr}(G)$ of G is non–identity. In the second case, it is not hard to see that G includes a non–identity normal locally finite subgroup. Clearly in every group G the subgroup $\mathbf{Lfr}(G)$ generated by all normal locally finite subgroups is the largest normal locally finite subgroup (the *locally finite radical*). Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors. A group G is called *locally generalized radical group*, if every finitely generated subgroup is generalized radical. The class of locally radical group is very large, in particular, it includes all locally finite groups and all locally soluble groups.

The main result is a following.

Theorem 1. *Let G be a locally generalized radical group, A a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module, and $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Suppose that G is not finitely generated, $G \neq D$ and $C_G(A) = \langle 1 \rangle$. Then G is a group of one of the following types.*

1. *G is a quasicyclic q-group for some prime q.*
2. *$G = Q \times \langle g \rangle$ where Q is a quasicyclic p-subgroup, g is a d-element and $g^d \in D$, p, d are prime (not necessary different).*
3. *G includes a normal divisible Chernikov p-subgroup Q, such that $G = Q\langle g \rangle$ where g is a d-element, p, d are prime (not necessary different). Moreover, G satisfies the following conditions:*
   - *(a) $g^d \in D$;*
   - *(b) Q is G-quasifinite;*
   - *(c) if $p = d$, then Q has special rank $d^{m-1}(d-1)$ where $d^m = |\langle g \rangle / C_{\langle g \rangle}(Q)|$;*
   - *(d) if $p \neq d$, then Q has special rank $\mathbf{o}(p, d^m)$ where again $d^m = |\langle g \rangle / C_{\langle g \rangle}(Q)|$ and $\mathbf{o}(p, d^m)$ is the order of p modulo d^m.*

Furthermore, for the types 2, 3 $A(\omega \mathbb{Z}_{p^\infty} D)$ is a Chernikov p-subgroup.

Here ωRG be the *augmentation ideal* of the group ring RG, the two-sided ideal of RG generated by all elements $g - 1$, $g \in G$.

Recall also that an abelian normal subgroup A of a group G is called *G-quasifinite* if every proper G-invariant subgroup of A is finite. Clearly that in this case either A is a union of its finite G-invariant subgroups or A includes a finite G-invariant subgroup B such that the factor A/B is a G-chief. At the end of the article, we provide the examples showing that all the situations that arise in the theorem can be realized.

1 Some preliminary results

Let R be a ring and $\mathfrak{M}$ a class of R-modules. Then $\mathfrak{M}$ is said to be a *formation* if it satisfies the following conditions:

F1. if $A \in \mathfrak{M}$ and B is an R-submodule of A, then $A/B \in \mathfrak{M}$;

F2. if A is an R-module and $B_1, \dots, B_k$ are R-submodules of A such that $A/B_j \in \mathfrak{M}$, $1 \leq j \leq k$, then $A/(B_1 \cap \dots \cap B_k) \in \mathfrak{M}$.

Lemma 1. *Let R be a ring, $\mathfrak{M}$ a formation of R-modules, G a group and A an RG-module.*

(i) If L, H are subgroups of G such that $L \leq H$ and $A/C_A(H) \in \mathfrak{M}$, then $A/C_A(L) \in \mathfrak{M}$.

(ii) If L, H are subgroups of G whose cocentralizers belong to $\mathfrak{M}$, then $A/C_A(\langle H, L \rangle) \in \mathfrak{M}$.

Proof. The inclusion $L \leq H$ implies that $C_A(L) \geq C_A(H)$. Since $A/C_A(H) \in \mathfrak{M}$ and $\mathfrak{M}$ is a formation, $A/C_A(L) \in \mathfrak{M}$. Clearly $C_A(\langle H, L \rangle) \leq C_A(H) \cap C_A(L)$. Since $\mathfrak{M}$ is a formation, $A/(C_A(H) \cap C_A(L)) \in \mathfrak{M}$. Then we have $A/C_A(\langle H, L \rangle) \in \mathfrak{M}$. □

Lemma 2. *Let R be a ring, $\mathfrak{M}$ a formation of R-modules, G a group and A an RG-module. Then*

$$\mathbf{Coc}_{\mathfrak{M}}(G) = \{x \in G \mid A/C_A(x) \in \mathfrak{M}\}$$

is a normal subgroup of G.

Proof. By Lemma 1 $\mathbf{Coc}_{\mathfrak{M}}(G)$ is a subgroup of G. Now let $x \in \mathbf{Coc}_{\mathfrak{M}}(G)$, $g \in G$. Then $C_A(x^g) = C_A(x)g$. Since the mapping $a \mapsto ag$, $a \in A$, is R-linear,

$$A/C_A(x) \cong_R Ag/C_A(x)g = A/C_A(x)g = A/C_A(x^g),$$

which shows that $A/C_A(x^g) \in \mathfrak{M}$, and hence $x^g \in \mathbf{Coc}_{\mathfrak{M}}(G)$. □

Clearly the class of minimax modules over an integral domain R is a formation and so we obtain the following result.

Corollary 1. *Let R be a ring, G a group and A an RG-module.*

(i) L, H are subgroups of G such that $L \leq H$ and a factor-module $A/C_A(H)$ is minimax, then $A/C_A(L)$ is also minimax.

(ii) If L, H are subgroups of G whose cocentralizers are minimax, then $A/C_A(\langle H, L \rangle)$ is also minimax.

Corollary 2. *Let R be a ring, G a group and A an RG-module. Then*

$$\mathbf{Coc}_{R-mmx}(G) = \{x \in G \mid A/C_A(x) \text{is minimax}\}$$

is a normal subgroup of G.

A group G is said to be *$\mathfrak{F}$-perfect* if G does not include proper subgroups of finite index.

Lemma 3. *Let G be a locally generalized radical group and A be a $\mathbb{Z}_{p^\infty}G$-module. Suppose that A includes a $\mathbb{Z}_{p^\infty}$–minimax $\mathbb{Z}_{p^\infty}G$-submodule B, which is minimax. Then the following assertions hold:*

(i) $G/C_G(B)$ is soluble-by-finite;

(ii) if $G/C_G(B)$ is periodic, then it is nilpotent-by-finite;

(iii) if $G/C_G(B)$ is $\mathfrak{F}$-perfect and periodic, then it is abelian; moreover $[[B, G], G] = \langle 0 \rangle$.

Proof. Without loss of generality we can suppose that $C_G(B) = \langle 1 \rangle$. Since B is minimax, it has a series of G-invariant subgroups $\langle 0 \rangle \leq D \leq K \leq B$ where D is divisible Chernikov subgroup, K/D is finite, and B/K is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. In particular, the $\Pi(D) = \{p\}$. Clearly D is G-invariant. The factor-group $G/C_G(D)$ is isomorphic to a subgroup of $\mathbf{GL}_m(\mathbb{Q}_{p^\infty})$ where $\mathbb{Q}_{p^\infty}$ is the field of fractions of $\mathbb{Z}_{p^\infty}$ and m satisfies $q^m = |\mathbf{\Omega}_1(D)|$. Let $\mathbb{Q}_{p^\infty}$ be a field of fractions of $\mathbb{Z}_{p^\infty}$, then $G/C_G(D)$ is isomorphic to a subgroup of $\mathbf{GL}_m(\mathbb{Q}_{p^\infty})$. Note that $\mathbf{char}(\mathbb{Q}_{p^\infty}) = 0$. Being locally generalized radical, $G/C_G(D)$ does not include the non-cyclic free subgroup; thus an application of Tits Theorem (see, for example, [12, Corollary 10.17]) shows that $G/C_G(D)$ is soluble-by-finite. If G is periodic, then $G/C_G(D)$ is finite (see, for example, [12, Theorem 9.33]). Since K/D is finite, $G/C_G(K/D)$ is finite. Finally, $G/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{GL}_r(\mathbb{Q}_{p^\infty})$, where $\mathbf{r} = \mathbf{r}_{\mathbb{Z}_{p^\infty}}(B/K)$. Using again the fact that $G/C_G(A/K)$ does not include the non-cyclic free subgroup and Tits Theorem or Theorem 9.33 of the book [12] (for periodic G), we obtain that $G/C_G(B/K)$ is soluble-by-finite (respectively finite whenever G is periodic). Put

$$Z = C_G(D) \cap C_G(K/D) \cap C_G(B/K).$$

Then G/Z is embedded in $G/C_G(D) \cap G/C_G(K/D) \cap G/C_G(B/K)$, in particular, G/Z is soluble-by- finite (respectively finite).

If $x \in Z$, then x acts trivially in every factors of the series $\langle 0 \rangle \leq D \leq K \leq A$. By Kaloujnin's theorem [7] Z is nilpotent. It follows that G is soluble-by-finite (respectively nilpotent-by-finite).

Suppose now that G is an $\mathfrak{F}$-perfect group. Again consider the series of G-invariant subgroups $\langle 0 \rangle \leq K \leq B$. Being abelian and Chernikov, K is a union of the ascending series

$$\langle 0 \rangle = K_0 \leq K_1 \leq ... \leq K_n \leq K_{n+1} \leq ...$$

of G-invariant finite subgroups K_n, $n \in \mathbb{N}$. Then the factor-group $G/C_G(K_n)$ is finite for every $n \in \mathbb{N}$. Since G is $\mathfrak{F}$-perfect, $G = C_G(K_n)$ for each $n \in \mathbb{N}$. The equality $K = \bigcup_{n\in\mathbb{N}} K_n$ implies that $G = C_G(K)$. As proved above, since $G/C_G(B/K)$ is soluble–by–finite and $\mathfrak{F}$–perfect, it is soluble. Then $G/C_G(B/K)$ includes normal subgroups U, V such that $C_G(B/K) \leq U \leq V$, $U/C_G(B/K)$ is isomorphic to a subgroup of $\mathbf{UT}_\mathbf{r}(\mathbb{Q}_{p^\infty})$, V/U includes a free abelian subgroup of finite index [1, Theorem 2]. Since $G/C_G(B/K)$ is $\mathfrak{F}$-perfect, it follows that $G/C_G(B/K)$ is torsion-free. Then $G/C_G(B/K)$ must be identity, because it is periodic. In other words, $G = C_G(B/K)$. Hence G acts trivially in every factors of a series $\langle 0 \rangle \leq K \leq B$, so that $[[B,G],G] = \langle 0 \rangle$, and using again Kaloujnin's theorem [7], we obtain that G is abelian. □

Lemma 4. *Let G be a Chernikov q-group and A a $\mathbb{Z}_{p^\infty}G$-module. If $A/C_A(G)$ is minimax (as a $\mathbb{Z}_{p^\infty}$-module), then the additive group of $A(\omega\mathbb{Z}_{p^\infty}G)$ is a Chernikov p-subgroup and $q = p$.*

Proof. For each element x of G consider the mapping $\delta_x : A \to A$, defined by the rule $\delta_x(a) = a(x-1)$, $a \in A$. Clearly this mapping is a $\mathbb{Z}_{p^\infty}$-endomorphism of A, $\mathbf{Ker}(\delta_x) = C_A(x)$ and $\mathbf{Im}(\delta_x) = A(\omega\mathbb{Z}_{p^\infty}\langle x \rangle) = A(x-1)$. Hence

$$A(x-1) = \mathbf{Im}(\delta_x) \cong_{\mathbb{Z}_{p^\infty}} A/\mathbf{Ker}(\delta_x) = A/C_A(x).$$

Since $A/C_A(G)$ is minimax, it has finite special rank r for some positive integer r. An inclusion $C_A(G) \leq C_A(x)$ follows that $A/C_A(x)$ has a special rank at most r. Then $\mathbf{r}(A(x-1)) \leq r$.

Let k be a positive integer such that $|\Omega_1(G)| = q^k$. Then G has an ascending series of finite subgroups

$$L_1 = \Omega_1(G) \leq L_2 \leq \ldots \leq L_n \leq L_{n+1} \leq \ldots$$

such that $L_n = \mathbf{Dr}_{1\leq j\leq k}\langle x_{n_j} \rangle$, where $|x_{n_j}| \leq q^n$ for each j, and $G = \bigcup_{n\in\mathbb{N}} L_n$. The equality

$$A(\omega\mathbb{Z}_{p^\infty}L_n) = A(\omega\mathbb{Z}_{p^\infty}\langle x_{n_1}\rangle) + \ldots + A(\omega\mathbb{Z}_{p^\infty}\langle x_{n_k}\rangle) = A(x_{n_1}-1) + \ldots + A(x_{n_k}-1)$$

together with $\mathbf{r}(A(x_{n_j}-1)) \leq r$, $1 \leq j \leq k$, shows that $\mathbf{r}(A(\omega\mathbb{Z}L_n)) \leq rk$ for every $n \in \mathbb{N}$. Since $G = \bigcup_{n\in\mathbb{N}} L_n$ we have $A(\omega\mathbb{Z}_{p^\infty}G) = \bigcup_{n\in\mathbb{N}} A(\omega\mathbb{Z}_{p^\infty}L_n)$, moreover $L_n \leq L_{n+1}$ implies that $A(\omega\mathbb{Z}_{p^\infty}L_n) \leq A(\omega\mathbb{Z}_{p^\infty}L_{n+1})$ for every $n \in \mathbb{N}$. Let B be an arbitrary finitely generated subgroup of $A(\omega\mathbb{Z}_{p^\infty}G)$. Then there exists a positive integer m such that $B \leq A(\omega\mathbb{Z}_{p^\infty}L_m)$. By proved above B has at most rk generators. It follows that $A(\omega\mathbb{Z}_{p^\infty}G)$ has a finite special rank at most rk.

Let Q be the divisible part of G. Since $A/C_A(Q)$ is minimax, A has a series of $\mathbb{Z}_{p^\infty}G$-submodules $C_A(Q) = C \leq T \leq A$ where $T/C = \mathbf{Tor}(A/C)$ is a Chernikov group and A/T is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. Repeating the final part of the proof of Lemma 3, we obtain that $Q = C_Q(T)$ and $Q = C_Q(A/T)$.

Let a be an arbitrary element of T. Consider the mapping $\gamma_a : Q \to A(\omega\mathbb{Z}_{p^\infty}Q)$, defined by the rule $\gamma_a(x) = a(x-1)$. By $(x-1)(y-1) = (xy-1) - (x-1) - (y-1)$. We have $a(xy-1) = a(x-1) + a(y-1) + a(x-1)(y-1) = a(x-1) + a(y-1)$. An equality $Q = C_Q(T)$ implies that $a(x-1)(y-1) = 0$. In other words, $\gamma_a(xy) = \gamma_a(x) + \gamma_a(y)$, thus γ_a is a homomorphism. Furthermore, $\mathbf{Ker}(\gamma_a) = C_Q(a)$ and $\mathbf{Im}(\gamma_a) = \langle a \rangle(\omega\mathbb{Z}_{p^\infty}Q) = [a,Q]$, so that $[a,Q] \cong Q/C_Q(a)$. It follows that if $[a,Q] \neq \langle 0 \rangle$, then it is a divisible Chernikov subgroup and $\Pi([a,Q]) \subseteq \Pi(Q) = \{q\}$. Since it is valid for every $a \in T$, $T(\omega\mathbb{Z}_{p^\infty}Q)$ is a divisible subgroup (if it is non-trivial) and $\Pi(T(\omega\mathbb{Z}_{p^\infty}Q)) \subseteq \Pi(Q) = \{q\}$. By proved above, $T(\omega\mathbb{Z}_{p^\infty}Q)$ has finite special rank, and therefore $T(\omega\mathbb{Z}_{p^\infty}Q)$ is a Chernikov subgroup.

Consider now the factor-module A/V where $V = T(\omega\mathbb{Z}_{p^\infty}Q)$. Then the inclusion $T/V \leq C_{A/V}(Q)$ implies that $(A/V)(\omega\mathbb{Z}_{p^\infty}Q) \leq T/V$. Using the above arguments, we obtain that $(A/V)(\omega\mathbb{Z}_{p^\infty}Q)$ is a Chernikov divisible group such that $\Pi((A/V)(\omega\mathbb{Z}_{p^\infty}Q)) \subseteq \Pi(Q)$. We have

$$(A/V)(\omega\mathbb{Z}_{p^\infty}Q) = (A(\omega\mathbb{Z}_{p^\infty}Q) + V)/V = (A(\omega\mathbb{Z}_{p^\infty}Q) + T(\omega\mathbb{Z}_{p^\infty}Q))/(T(\omega\mathbb{Z}_{p^\infty}Q),$$

which follows that $A(\omega\mathbb{Z}_{p^\infty}Q)$ is a Chernikov divisible subgroup such that $\Pi(A(\omega\mathbb{Z}_{p^\infty}Q)) \subseteq \Pi(Q)$.

Let $M = A(\omega\mathbb{Z}_{p^\infty}Q)$, then $Q \leq C_G(A/M)$, in particular, $G/C_G(A/M)$ is finite. By proved above $(A/M)(\omega\mathbb{Z}_{p^\infty}G)$ has finite special rank. Using the above arguments, we obtain that $\langle a+M \rangle(\omega\mathbb{Z}_{p^\infty}G)$ is a finite group and $\Pi(\langle a+M \rangle(\omega\mathbb{Z}_{p^\infty}G)) \subseteq \Pi(G) = \{q\}$ for every element $a \in A$. The finiteness of $\Pi(G)$ implies that $(A/M)(\omega\mathbb{Z}_{p^\infty}G)$ is a Chernikov subgroup of A/M and $\Pi((A/M)(\omega\mathbb{Z}_{p^\infty}G)) \subseteq \Pi(G) = \{q\}$. Hence $A(\omega\mathbb{Z}_{p^\infty}G)$ is Chernikov and $\Pi(A(\omega\mathbb{Z}_{p^\infty}G)) \subseteq \Pi(G) = \{q\}$ but $\Pi(A(\omega\mathbb{Z}_{p^\infty}G)) \subseteq \{p\}$ so we have $q = p$. □

Corollary 3. *Let G be a group and A a $\mathbb{Z}_{p^\infty}G$-module. If $A/C_A(G)$ is minimax as $\mathbb{Z}_{p^\infty}$-module, then every locally generalized radical subgroup of $G/C_G(A)$ is soluble-by- finite, and every periodic subgroup of $G/C_G(A)$ is nilpotent-by-finite.*

Proof. Indeed, Lemma 3 shows that $G/C_G(A/C_A(G))$ is soluble-by-finite. Every element $x \in C_G(A/C_A(G))$ acts trivially in the factors of the series $\langle 0\rangle \leq C_A(G) \leq A$. It follows that $C_G(A/C_A(G))$ is abelian. Suppose now that $H/C_G(A)$ is a periodic subgroup of $G/C_G(A)$. Since $A/C_A(G)$ is minimax, A has a series of H-invariant subgroups

$$\langle 0\rangle \leq C_A(G) \leq D \leq K \leq A,$$

where $D/C_A(G)$ is a divisible Chernikov subgroup, K/D is finite and A/K is torsion-free and has finite $\mathbb{Z}_{p^\infty}$-rank. In Lemma 3 we have already proved that $G/C_G(D/C_A(G))$, $G/C_G(K/D)$ and $G/C_G(A/K)$ are finite. Let $Z = C_G(D/C_A(G)) \cap C_G(K/D) \cap C_G(A/K)$. Then G/Z is finite. If $x \in Z$, then x acts trivially in every factors of a series $\langle 0\rangle \leq C_A(G) \leq D \leq K \leq A$. By Kaloujnin's theorem [7] Z is nilpotent. □

Let G be a generalized radical group and let R_1 be a normal subgroup of G, satisfying the following conditions: R_1 is radical, G/R_1 does not include the non-trivial locally nilpotent normal subgroups. Then G/R_1 must include a non-trivial normal locally finite subgroup. It follows that the locally finite radical R_2/R_1 is non-trivial. If we suppose that G/R_2 includes a non-trivial normal locally finite subgroup L/R_2, then L/R_1 is also locally finite, which contradicts the choice of R_2. This contradiction shows that G/R_2 does not include a non-identity normal locally finite subgroup, and therefore it must include a non-identity normal locally nilpotent subgroup. Let R_3/R_2 be a normal subgroup of G/R_2 satisfying the following conditions: R_3/R_2 is radical, G/R_3 does not include non-identity locally nilpotent normal subgroups. Using similar arguments, we construct the ascending series of normal subgroups

$$\langle 1\rangle = R_0 \leq R_1 \leq \ldots R_\alpha \leq R_{\alpha+1} \leq \ldots R_\gamma = G,$$

whose factors are radical or locally finite, and if $R_{\alpha+1}/R_\alpha$ is radical (respectively locally finite), then $R_{\alpha+2}/R_{\alpha+1}$ is locally finite (respectively radical).

This series is called a *standard series* of a generalized radical group G.

Lemma 5. *Let G be a group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Then every proper generalized radical subgroup of $G/C_G(A)$ is soluble-by-finite.*

Proof. Again we will suppose that $C_G(A) = \langle 1\rangle$. Let L be an arbitrary proper generalized radical subgroup of G. Let

$$\langle 1\rangle = R_0 \leq R_1 \leq \ldots R_\alpha \leq R_{\alpha+1} \leq \ldots R_\gamma = L,$$

be a standard series of L. Suppose that $\gamma \geq \omega$ (ω is the first infinite ordinal) and consider the subgroup R_ω. Assume that R_ω is finitely generated, that is $R_\omega = \langle u_1, \ldots, u_t\rangle$ for some elements $u_1, \ldots, u_t$. The equality $R_\omega = \bigcup_{n\in\mathbb{N}} R_n$ shows that there exists a positive integer m such that $u_1, \ldots, u_t \in R_m$. But in this case, $R_\omega = R_m$ and we obtain a contradiction. This contradiction shows that R_ω is not finitely generated. It follows that $A/C_A(R_\omega)$ is minimax. Corollary 3 shows that R_ω is soluble-by-finite. In this case $R_\omega = R_2$ and we again obtain a contradiction. This contradiction shows that $\gamma = k$ for some positive integer.

Now we will use induction by k for a proof of our assertion. Consider the subgroup R_1. Then either R_1 is radical or locally finite. If R_1 is not finitely generated, then $A/C_A(R_\omega)$ is minimax. Corollary 3 shows that R_1 is soluble-by-finite. Suppose that R_1 is finitely generated. If R_1 is locally finite, then it is finite. Therefore assume that R_1 is radical. Let

$$\langle 1\rangle = V_0 \leq V_1 \leq \ldots V_\alpha \leq V_{\alpha+1} \leq \ldots V_\eta = R_1,$$

be an ascending series of R_1 where $V_{\alpha+1}/V_\alpha$ is the locally nilpotent radical of R_1/V_α, $\alpha < \eta$. Using the above arguments we obtain that $\eta = d$ for some positive integer d. Let m be a number such that all factors $V_{m+1}/V_m, V_{m+2}/V_{m+1}, \ldots, V_d/V_{d-1}$ are finitely generated. Since they are locally nilpotent, they must be polycyclic. It follows that V_d/V_m is polycyclic. In particular if every subgroup V_j is finitely generated, $1 \leq j \leq d$, then R_1 is polycyclic. Therefore assume that there is a positive integer s such that V_s is not finitely generated, but a subgroup V_j is finitely generated for all $j > s$. Then $A/C_A(V_s)$ is minimax and Corollary 3 yields that V_s is soluble. In this case R_1/V_s is polycyclic, so that R_1 is soluble.

Suppose that we have already proved that all subgroups $R_1, R_2, \ldots, R_{k-1}$ are soluble-by-finite. Repeating the above arguments, we obtain that and R_k is soluble-by-finite, and the result is proved. □

Lemma 6. *Let G be a group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. If H is a proper subgroup of G and $\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ does not include H, then H is finitely generated.*

Proof. Indeed if we suppose that H is not finitely generated, then $A/C_A(H)$ is minimax. Corollary 1 shows that $A/C_A(h)$ is minimax for each element $h \in H$. It follows that $H \leq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, and we obtain a contradiction with the choice of H. □

2 Proofs of the main results

Proposition 1. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. If $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is not finitely generated, then $G/C_G(A)$ is a quasicyclic q-group for some prime q.*

Proof. Again suppose that $C_G(A) = \langle 1\rangle$. Let $M = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Let H be a proper subgroup of G. If M does not include H, then Lemma 6 shows that H is finitely generated. In particular if $M \leq H$, then H/M is finitely generated. In other words, every proper subgroup of G/M is finitely generated. By [8, Proposition 2.7] G/M is a quasicyclic q-group for some prime q.

Let L/M be a proper subgroup of G/M, then L/M is a finite cyclic subgroup. An application of Lemma 6 shows that the subgroup L is finitely generated. The finiteness of index $|L : M|$ implies that M is finitely generated (see, for example, [5, Corollary 7.2.1]). Using Lemma 5 we obtain that M is soluble-by-finite. Then M includes a maximal normal soluble subgroup S such that M/S is finite. It is not hard to see, that S is G-invariant. Let $D = S'$, then M/D is abelian-by-finite and finitely generated, therefore it is noetherian. Let V/D be a proper subgroup of G/D. If M/D does not include V/D, then M does not include V, and as above, V is finitely generated. Then V/D is also finitely generated. If $V/D \leq M/D$, then again V/D is finitely generated. Thus every proper subgroup of G/D is finitely generated, and application of [8, Proposition 2.7] shows that G/D is a quasicyclic group. Since M/D is a proper subgroup of G/D, M/D is a finite cyclic subgroup. Suppose that $D \neq \langle 1\rangle$, then $K = D' \neq D$. Repeating the above arguments, we obtain that G/K is a quasicyclic group. In particular, it is abelian.

Then S/K is abelian, which follows that $K \geq S' = D$. This contradiction shows that $D = \langle 1 \rangle$, so that G is a quasicyclic group. □

Lemma 7. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Suppose that $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, G is not finitely generated, and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is finitely generated. Then G is soluble and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is a group of a prime order q.*

Proof. Again suppose that $C_G(A) = \langle 1 \rangle$. Let $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Since G/D is finitely generated, $G = \langle M, D \rangle$ for some finite subset M of G. We may suppose that M is minimal finite set with this property, that is $G \neq \langle S, D \rangle$ for each proper subset S of M. Now suppose that $|M| \geq 2$. Then M includes two proper subsets X, Y such that $M = X \cup Y$. By the choice of M, the subgroups $\langle X, D \rangle$ and $\langle Y, D \rangle$ are proper and also $\langle X, D \rangle \neq D$, $\langle Y, D \rangle \neq D$. By Lemma 6 both subgroups $\langle X, D \rangle$ and $\langle Y, D \rangle$ are finitely generated. An equality $X \cup Y = M$ implies that $G = \langle X, Y, D \rangle$ is finitely generated. This contradiction shows that $|M| = 1$. In other words, G/D is cyclic. Suppose that $|G/D|$ is not a prime. Then G includes a proper subgroup B such that $D \leq B$, $B \neq D$, and G/B has a prime order. Using Lemma 6 we obtain that B is finitely generated. The finiteness of G/B gives that G is finitely generated. This final contradiction proves that G/D has a prime order. Choose an element g such that $G = \langle g, D \rangle$.

Since G is not finitely generated, D cannot be finitely generated. Using Lemma 5, we obtain that D is soluble-by-finite. Let S be a maximal normal soluble subgroup of D having finite index. Suppose that $D \neq S$. Clearly S is G-invariant. Since D/S is finite and non-soluble, $S\langle g^q \rangle \neq D$. It follows that $S\langle g \rangle$ is a proper subgroup of G. Since D does not include $S\langle g \rangle$, $S\langle g \rangle$ is finitely generated by Lemma 6. Then $S\langle g^q \rangle$ is finitely generated (see, for example, [5, Corollary 7.2.1]). Since the index $|D : S|$ is finite, D is finitely generated. This contradiction shows that D is soluble. Hence G is soluble. □

Let K be a finite group. We have $|K| = p_1^{t_1} \dots p_s^{t_s}$ where $p_1, \dots, p_s$ are primes and $p_m \neq p_j$ whenever $m \neq j$. Put $\Pi(K) = \{p_1, \dots, p_s\}$.

Corollary 4. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module and $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. Suppose that $G \neq D$, G is not finitely generated and G/D is finitely generated. Let g be an element of G with a property $G = \langle g \rangle D$. If H is a normal subgroup of G, having finite index, then $H\langle g \rangle = G$. Moreover, G/H is a q-group.*

Proof. If we assume that $H\langle g \rangle$ is a proper subgroup of G, then the choice of g yields that D does not include $H\langle g \rangle$. By Lemma 6, $H\langle g \rangle$ is finitely generated. Since $H\langle g \rangle$ has finite index, G must be finitely generated. This contradiction shows that $H\langle g \rangle = G$.

Suppose that $\Pi(G/H) \neq \{q\}$. Let P/H be a Sylow q-subgroup of G/H. Then P/H is a proper subgroup of G/H. Since P has finite index in G, P is not finitely generated. Then $A/C_A(P)$ is minimax. It follows that $P \leq D$. On the other hand, G/D is a non-trivial q-group and therefore D cannot include P. This contradiction proves that G/H is a q-group. □

Let G be a group, denote by $\mathbf{Tor}(G)$ the maximal normal periodic subgroup of G (periodic part of G).

Proposition 2. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Suppose that $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, G is not finitely generated and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is finitely generated. If G/G' is infinite, then $G = Q \times \langle g \rangle$ where Q is a quasicyclic p-subgroup, g is a d-element and $g^d \in \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, where p, d are primes (not necessary different).*

Proof. As usual we suppose that $C_G(A) = \langle 1 \rangle$. Let $D = \mathbf{Coc}_{\mathbb{Z}-mmx}(G)$. By Lemma 7, G is soluble and G/D is a group of a prime order q. Choose an element g such that $G = \langle g, D \rangle$.

Put $K = G'$, then $K \leq D$. Suppose that $K\langle g \rangle = G$. Since G/K is infinite and $G/K = K\langle g \rangle/K \cong \langle g \rangle/(\langle g \rangle \cap K)$ we obtain that gK has infinite order. Let r_1, r_2 be two distinct primes. Then $K\langle g^{r_j} \rangle$ is a proper subgroup of G. Since it has finite index in G, $K\langle g^{r_j} \rangle$ is not finitely generated. It follows that $A/C_A(K\langle g^{r_j} \rangle)$ is minimax for every $j \in \{1, 2\}$. Since $r_1 \neq r_2$ we have $\langle g \rangle = \langle g^{r_1} \rangle \langle g^{r_2} \rangle$. Corollary 1 shows that $A/C_A(\langle g \rangle)$ is minimax, that is $g \in D$, and we obtain a contradiction with the choice of g. It shows that $K\langle g \rangle$ is a proper subgroup of G.

Now let $Z/(K\langle g \rangle)$ be a proper subgroup of $G/(K\langle g \rangle)$. Then D does not include Z and hence Lemma 6 shows that Z is finitely generated. If we assume that Z has finite index in G, then G must be finitely generated, so we obtain a contradiction. This contradiction shows that the factor-group $G/(K\langle g \rangle)$ is $\mathfrak{F}$-perfect. Then $G/(K\langle g \rangle)$ includes a subgroup $P/(K\langle g \rangle)$ such that G/P is a quasicyclic d-group for some prime d. Since $g \in P$, we have that D does not include P. By Lemma 6, P is finitely generated. It follows that G/K is an abelian minimax group. Suppose that $\mathbf{Tor}(G/K) \neq G/K$. Then $T/K = \mathbf{Tor}(D/K) \neq D/K$. Put

$$\pi = \{r \mid r \text{ is a prime such that } D/T \neq (D/T)^r\}.$$

Since D/T is torsion-free and minimax, the set π is infinite. Therefore we can choose a prime r such that $r \neq q$ and $r \in \pi$. Let $L/T = (D/T)^r$, then D/L is a non-identity elementary abelian r-group. By the choice of L, $\Pi(G/L) = \{r, q\}$ and this contradicts Corollary 4. Hence we have that G/K is periodic. In this case, P/K is finite, so that G/K is a Chernikov group. Let Q/K be the divisible part of G/K. Since $Q/K \cong G/P$, Q/K is a quasicyclic q-subgroup. Since Q has finite index in G, Corollary 4 shows that $G = Q\langle g \rangle$ and G/Q is a q-group. It follows that $G/K = Q/K \times \langle gK \rangle$ (see [2, Theorem 21.2]). Moreover, by Lemma 4 Q is a p-group.

Suppose that $K \neq \langle 1 \rangle$. Then $L = K' \neq K$. We have already proved above that $K\langle g \rangle$ is a proper subgroup of G. Since D does not include $K\langle g \rangle$, Lemma 6 shows that $K\langle g \rangle$ is finitely generated. The fact that G/K is periodic implies that K has finite index in $K\langle g \rangle$. Then K is finitely generated (see, for example, [5, Corollary 7.2.1]). Thus K/L is a finitely generated abelian group. Then K/L includes a proper G-invariant subgroup V/L of finite index in K/L (this subgroup can be identity). Then G/V is a Chernikov group with finite derived subgroup. Let Q_1/V be the divisible part of G/V, then $Q_1/V \cong Q/K$, so that Q_1/V is a quasicyclic q-subgroup. Since $[G/V, G/V]$ is finite, $Q_1/V \leq \zeta(G/V)$. Since index $|G : Q_1|$ is finite, $G = Q_1\langle g \rangle$ by Corollary 4. This equality together with the inclusion $Q_1/V \leq \zeta(G/V)$ implies that G/V is abelian. But in this case $K \leq V$, and this contradicts with the choice of V. Consequently we have $K = \langle 1 \rangle$. So Q is a proper subgroup of G which is quasicyclic q-group, than by Lemma 4 Q is a p-group. □

Proposition 3. *Let G be a locally generalized radical group and let A be a minimax-antifinitary $\mathbb{Z}_{p^\infty}G$-module. Suppose that $G \neq \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$, G is not finitely generated and $G/\mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ is finitely generated. If G/G' is finite, then G includes a normal divisible Chernikov p-subgroup Q, such that $G = Q\langle g \rangle$ where g is a d-element, $g^d \in \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$ and p, d are primes (not necessary different). Moreover, the subgroup Q is G-quasifinite.*

Proof. As usual we suppose that $C_G(A) = \langle 1 \rangle$. Let $D = \mathbf{Coc}_{\mathbb{Z}_{p^\infty}-mmx}(G)$. By Lemma 7, G is soluble and G/D is a group of a prime order d. Choose an element g such that $G = \langle g, D \rangle$.

Put $K = G'$. Since G/K is finite, Corollary 4 shows that $G = K\langle g \rangle$ and G/K is a d-group. It follows that K is not finitely generated.

Since G is not finitely generated and soluble, $L = K'$ is a proper subgroup of K. If we suppose that $\langle g, L \rangle = G$, then $G/L = \langle g \rangle L/L \cong \langle g \rangle/(\langle g \rangle \cap L)$ is abelian. It follows that $K \leq L$, and we obtain a contradiction. Thus $\langle g, L \rangle$ is a proper subgroup of G. If we suppose that G/L is finite, then Corollary 4 shows that $G = L\langle g \rangle$. Hence G/L is infinite, i.e. K/L is infinite. As we noted above, $\langle L, g \rangle$ is a proper subgroup of G. Since D does not include $\langle L, g \rangle$, $\langle L, g \rangle$ is finitely generated by Lemma 6. A subgroup $\langle g \rangle \cap K$ is cyclic, so that $\langle g \rangle \cap K = \langle v \rangle$ for some $v \in \langle g \rangle$. Then we have

$$K \cap (L\langle g \rangle) = L(K \cap \langle g \rangle) = L\langle v \rangle.$$

Clearly $L\langle v \rangle$ is a G-invariant subgroup of K. Furthermore, $|\langle L, g \rangle : L\langle v \rangle| \leq |G : D| = d$. It follows that $\langle L, v \rangle$ is finitely generated (see, for example, [5, Corollary 7.2.1]). If we suppose that $K/(L\langle v \rangle)$ is finitely generated, then K is finitely generated. This contradiction shows that $K/(L\langle v \rangle)$ is not finitely generated.

Let $Z/(L\langle v \rangle)$ be a proper G-invariant subgroup of $K/(L\langle v \rangle)$, then we have $Z\langle g \rangle \cap K = X(\langle g \rangle \cap K) = Z\langle v \rangle = Z$. It follows that $Z\langle g \rangle$ is a proper subgroup of G. Since D does not include $Z\langle g \rangle$, $Z\langle g \rangle$ is finitely generated by Lemma 6.

Assume that $K/(L\langle v \rangle)$ includes a proper subgroup $U/\langle L, v \rangle$ of finite index. Then $|G : U|$ is finite, so that $U_1 = \mathbf{Core}_G(U)$ has finite index in G. By above proved $U_1\langle g \rangle$ is finitely generated. Finiteness of $|G : U_1|$ implies that G is finitely generated. This contradiction shows that $K/(L\langle v \rangle)$ is $\mathfrak{F}$-perfect. Then $K/(L\langle v \rangle)$ includes a subgroup $P/(L\langle v \rangle)$ such that K/P is a quasicyclic q-group for some prime q. We remark that $K/P^x = K^x/P^x \cong K/P$, i.e. K/P^x is a quasicyclic q- group for all $x \in G$. Finiteness of G/K implies that the family $\{P^x \mid x \in G\}$ is finite. Let $\{P^x \mid x \in G\} = \{P_1, P_2, \ldots, P_m\}$ where $P_1 = P$. Then the embedding

$$K/\mathbf{Core}_G(P) \hookrightarrow G/P_1 \times G/P_2 \times \ldots \times G/P_m,$$

shows that $K/\mathbf{Core}_G(P)$ is a Chernikov q-group. Since $K/\mathbf{Core}_G(P)$ is $\mathfrak{F}$- perfect, it is divisible. Since $\langle L, v \rangle \leq P$ and $\langle L, v \rangle$ is G-invariant, $\langle L, v \rangle \leq C = \mathbf{Core}_G(P)$. By proved above, C is finitely generated. In particular, C/L is an abelian finitely generated group, so that K/L is an abelian minimax group. Suppose that $\mathbf{Tor}(K/L) = T/L \neq K/L$. Put

$$\pi = \{r \mid r \text{ is a prime such that } K/T \neq (K/T)^r\}.$$

Since K/T is torsion-free and minimax, the set π is infinite. Therefore we can choose a prime r such that $r \neq d$ and $r \in \pi$. Let $M/T = (K/T)^r$, then K/M is a non-identity elementary abelian r- group. Clearly a subgroup M is G-invariant. By the choice of M, $\Pi(G/M) = \{r, d\}$. This contradiction with Corollary 4 shows that K/L is periodic. In this case, C/L is finite, so that K/L is Chernikov. Let Q/L be a divisible part of K/L. The isomorphism $Q/L \cong K/C$ shows that Q/L is a q-subgroup. Since Q has finite index, an application of Corollary 4 shows that $G = Q\langle g \rangle$ and G/Q is a d-group.

Suppose that Q/L includes an infinite G-invariant subgroup Q_1/L and that $Q_1\langle g \rangle$ is finitely generated. Then $Q_1\langle g \rangle/L = (Q_1/L)\langle gL \rangle$ is also finitely generated. Now G/L is periodic, in particular, $\langle gL \rangle$ is finite. It follows that Q_1/L is finitely generated. On the other hand, Q_1/L is an infinite Chernikov group, therefore it cannot be finitely generated. This contradiction shows that $Q_1\langle g \rangle$ is not finitely generated. Then $A/C_A(Q_1\langle g \rangle)$ is minimax. Corollary 1 shows that $g \in D$. This contradiction shows that Q/L is G-quasifinite.

Suppose that $L \neq \langle 1 \rangle$. Then $V = L' \neq L$. We have already proved that $L\langle g \rangle$ is is finitely generated. The fact that G/L is periodic implies that L has finite index in $L\langle g \rangle$. Then L is finitely generated (see, for example, [5, Corollary 7.2.1]). Thus L/V is a finitely generated abelian group. Then L/V includes a proper G-invariant subgroup W/V of finite index in L/V (this subgroup can be identity). Then K/W is a Chernikov group, having finite derived subgroup. Let Q_2/W be the divisible part of K/W, then $Q_2/W \cong Q/L$, so that Q_2/W is a divisible Chernikov q-subgroup. Since $(K/W)'$ is finite, $Q_2/W \leq \zeta(K/W)$. Since index $|G : Q_2|$ is finite, $G = Q_2\langle g \rangle$ by Corollary 4. Then

$$K = K \cap (Q_2\langle g \rangle) = Q_2(K \cap \langle g \rangle) = Q_2\langle v \rangle.$$

It follows that K/Q_2 is cyclic. Then the inclusion $Q_2/W \leq \zeta(K/W)$ implies that K/W is abelian. But $L \leq W$, and this contradiction the choice of W. Consequently L is abelian. So Q is a proper subgroup of G which is Chernikov q-group, than by Lemma 4 Q is a p-group. □

Recall that a group G have *finite special rank* $\mathbf{r}(G) = r$ if every finitely generated sub group of G has at most r generators and there exists a finitely generated subgroup H of G such that H has exactly r generators. Therefore every abelian minimax group has finite special rank.

3 Proof of the Main Theorem

If G/D is not finitely generated, then Proposition 1 shows that G is a group of type (1).

Suppose now that G/D is finitely generated. Then Lemma 7 proves that G is soluble and G/D is a group of a prime order q. If we assume that G/G' is infinite, then Proposition 2 shows that G is a group of type (2).

Finally suppose that G/G' is finite. Then Proposition 3 shows that G includes a normal divisible Chernikov p-subgroup Q, such that $G = Q\langle g \rangle$ where g is a d-element, p, d are primes (not necessary different). Moreover, $g^d \in D$ and Q is G-quasi-finite. Finally, the assertion 3c follows from the results of Section 3 of the paper [21], and the assertion 3d follows from Theorem 3.4 of the paper [6].

Let G be a group of the type (2) or (3). Then $D = Q\langle g^d \rangle$ is a proper Chernikov subgroup of G, and hence it is not finitely generated. Then $A/C_A(D)$ is minimax and Lemma 4 proves the final assertion.

References

[1] Charin V.S. *On the automorphism groups of nilpotent groups*. Ukrain. Mat. Zh. 1954, **6** (3), 295–304. (in Ukrainian)

[2] Fuchs L. Infinite abelian groups. Academic Press, New York, 1970.

[3] Hall P. *Finiteness conditions for soluble groups*. Proc. London Math. Soc. (3) 1954, **3-4** (1), 419–436. doi:10.1112/plms/s3-4.1.419

[4] Hall P. *On the finiteness of certain soluble groups*. Proc.London Math. Soc. (3) 1959, **3-9** (4), 595–622. doi:10.1112/plms/s3-9.4.595

[5] Hall M. The theory of groups. Macmillan, New York, 1959.

[6] Hartley B. *A dual approach to Chernikov modules.* Math. Proc. Cambridge Philos. Soc. 1977, **82** (2), 215–239. doi:10.1017/S0305004100053834

[7] Kaloujnin L.A. *Über gewisse Beziehungen zwischen eine Gruppe und ihren Automorphismen.* Bericht Math. Tagung Berlin 1953, **4**, 164–172.

[8] Kurdachenko L.A., Muñoz-Escolano J.M., Otal J. *Antifinitary linear groups.* Forum Math. 2008, **20** (1), 27–44. doi:10.1515/FORUM.2008.002

[9] Kurdachenko L.A., Otal J., Subbotin I.Ya. Artinian modules over group rings. Birkhäuser, Basel, 2007.

[10] Kurdachenko L.A., Subbotin I.Ya., Chupordya V.A. *On bounded artinian-finitary modules.* Internat. J. Algebra Comput. 2007, **17** (4), 881–893. doi:10.1142/S021819670700386X

[11] Phillips R. *Finitary linear groups: a survey.* In: Finite and locally finite groups, NATO ASI Series, 471. Dordrecht, Kluver, 1995. 111–146.

[12] Wehrfritz B.A.F. Infinite linear groups. Springer, Berlin, 1973.

[13] Wehrfritz B.A.F. *Finite-finitary groups of automorphisms.* J. Algebra Appl. 2002, **1** (4), 375–389. doi:10.1142/S0219498802000318

[14] Wehrfritz B.A.F. *On generalized finitary groups.* J. Algebra 2002, **247** (2), 707–727. doi:10.1006/jabr.2001.9015

[15] Wehrfritz B.A.F. *Finitary automorphism groups over commutative rings.* J. Pure Appl. Algebra 2002, **172** (2-3), 337–346. doi:10.1016/S0022-4049(01)00142-6

[16] Wehrfritz B.A.F. *Finitary and artinian-finitary groups over the integers $\mathbb{Z}$.* Ukrainian Math. J. 2002, **54** (6), 924–936. doi:10.1023/A:1021756002353

[17] Wehrfritz B.A.F. *Artinian-finitary groups over commutative rings.* Illinois J. Math. 2003, **47** (1-2), 551–565.

[18] Wehrfritz B.A.F. *Finitary and artinian-finitary groups over commutative rings.* J. Group Theory 2004, **7** (2), 243–253.

[19] Wehrfritz B.A.F. *Artinian-finitary groups over commutative rings and non-commutative rings.* J. London Math. Soc. (2) 2004, **70** (2), 325–340. doi:10.1112/S0024610704005411

[20] Wehrfritz B.A.F. *Artinian-finitary groups are locally normal-finitary.* J. Algebra 2005, **287** (2), 417–431. doi:10.1016/j.jalgebra.2005.01.034

[21] Zaitsev D.I. On the complementations of the subgroups in extremal groups. In: Investigations of the groups on prescribed properties of subgroups. Inst. Math., Kiev, 1974.

Received 09.06.2014

Revised 02.03.2015

Чупордя В.А. *Про структуру деяких мінімаксно-антифінітарних модулів* // Карпатські матем. публ. — 2015. — Т.7, №1. — C. 120–132.

Нехай R — кільце, G — група. R-модуль A називається *мінімаксним* якщо A містить нетеровий підмодуль B такий, що A/B артіновий. Вивчаються $\mathbb{Z}_{p^\infty}G$-модулі A такі, що $A/C_A(H)$ є мінімаксним як $\mathbb{Z}_{p^\infty}$-модуль, для кожної власної підгрупи H, яка не є скінченно породженою.

Ключові слова і фрази: мінімаксний модуль, коцентралізатор, модуль над груповим кільцем, мінімаксно-антифінітарний RG-модуль, узагальнено радикальна група.

Науковий журнал

Карпатські Математичні Публікації

(свідоцтво про державну реєстрацію КВ № 14703-3674Р)

Том 7, №1
2015

Відповідальний за випуск д.ф.-м.н. Загороднюк А.В.

Комп'ютерна правка та макетування Івасюк І.Я.

Підписано до друку 29.06.2015 р. Формат 60×84/8.
Папір офсетний. Друк цифровий. Гарнітура URW Palladio L, Pazo Math
Умовн. друк. аркушів 15,6. Наклад 100 примірників.

Друк: пп Голіней О.М.
м. Івано-Франківськ, вул. Галицька, 128
тел. 0342 58 04 32